

# CHAPTER 3

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# KINEMATICS OF MECHANISMS

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### 3.1 DESIGN USE OF THE MECHANISMS SECTION

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The design process involves intuition, invention, synthesis, and analysis. Although no arbitrary rules can be given, the following design procedure is suggested:

1. Define the problem in terms of inputs, outputs, their time-displacement curves, sequencing, and interlocks.
2. Select a suitable mechanism, either from experience or with the help of the several available compilations of mechanisms, mechanical movements, and components (Sec. 3.8).
3. To aid systematic selection consider the creation of mechanisms by the separation of structure and function and, if necessary, modify the initial selection (Secs. 3.2 and 3.6).
4. Develop a first approximation to the mechanism proportions from known design requirements, layouts, geometry, velocity and acceleration analysis, and path-curvature considerations (Secs. 3.3 and 3.4).
5. Obtain a more precise dimensional synthesis, such as outlined in Sec. 3.5, possibly with the aid of computer programs, charts, diagrams, tables, and atlases (Secs. 3.5, 3.6, 3.7, and 3.9).
6. Complete the design by the methods outlined in Sec. 3.6 and check end results. Note that cams, power screws, and precision gearing are treated in Chaps. 14, 16, and 21, respectively.

### 3.2 BASIC CONCEPTS

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#### 3.2.1 Kinematic Elements

Mechanisms are often studied as though made up of rigid-body members, or “links,” connected to each other by rigid “kinematic elements” or “element pairs.” The nature and arrangement of the kinematic links and elements determine the kinematic properties of the mechanism.

If two mating elements are in surface contact, they are said to form a “lower pair”; element pairs with line or point contact form “higher pairs.” Three types of lower pairs permit relative motion of one degree of freedom ( $f = 1$ ), turning pairs, sliding pairs, and screw pairs. These and examples of higher pairs are shown in Fig. 3.1. Examples of element pairs whose relative motion possesses up to five degrees of freedom are shown in Fig. 3.2.

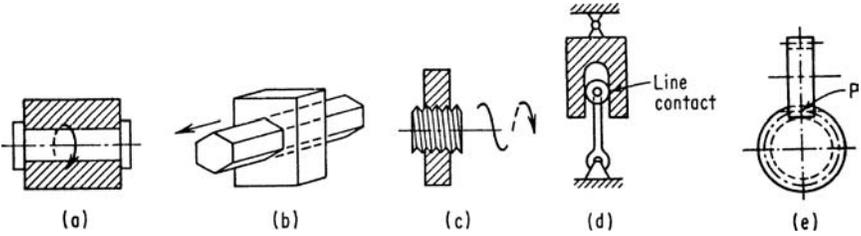


FIG. 3.1 Examples of kinematic-element pairs: lower pairs *a*, *b*, *c*, and higher pairs *d* and *e*. (*a*) Turning or revolute pair. (*b*) Sliding or prismatic pair. (*c*) Screw pair. (*d*) Roller in slot. (*e*) Helical gears at right angles.

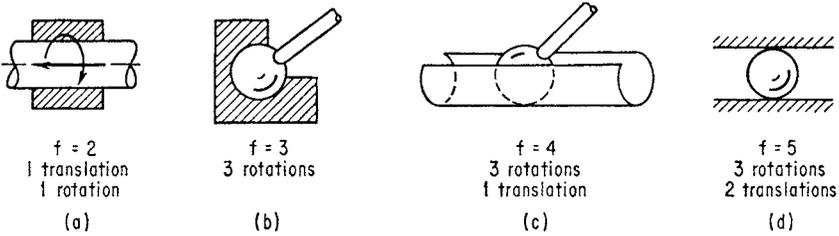


FIG. 3.2 Examples of elements pairs with  $f > 1$ . (*a*) Turn slide or cylindrical pair. (*b*) Ball joint or spherical pair. (*c*) Ball joint in cylindrical slide. (*d*) Ball between two planes. (Translational freedoms are in mutually perpendicular directions. Rotational freedoms are about mutually perpendicular axes.)

A link is called “binary,” “ternary,” or “*n*-nary” according to the number of element pairs connected to it, i.e., 2, 3, or *n*. A ternary link, pivoted as in Fig. 3.3*a* and *b*, is often called a “rocker” or a “bell crank,” according to whether  $\alpha$  is obtuse or acute.

A ternary link having three parallel turning-pair connections with coplanar axes, one of which is fixed, is called a “lever” when used to overcome a weight or resistance (Fig. 3.3*c*, *d*, and *e*). A link without fixed elements is called a “floating link.”

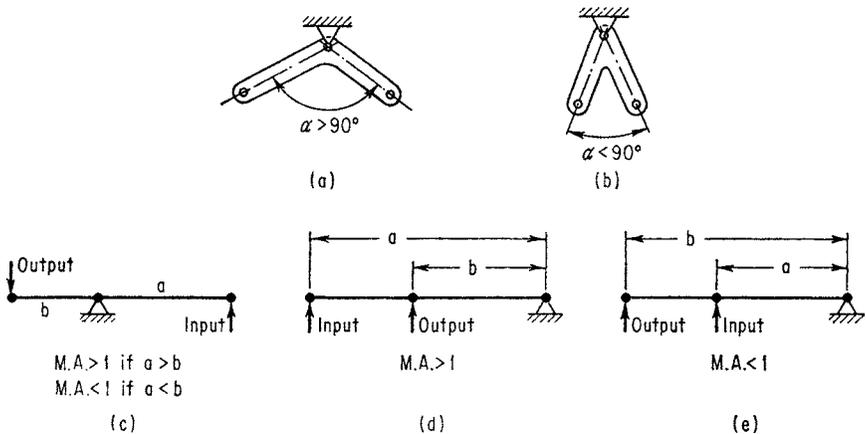


FIG. 3.3 Links and levers. (*a*) Rocker (ternary link). (*b*) Bell crank (ternary link). (*c*) First-class lever. (*d*) Second-class lever. (*e*) Third-class lever.

Mechanisms consisting of a chain of rigid links (one of which, the “frame,” is considered fixed) are said to be closed by “pair closure” if all element pairs are constrained by material boundaries. All others, such as may involve springs or body forces for chain closure, are said to be closed by means of “force closure.” In the latter, nonrigid elements may be included in the chain.

### 3.2.2 Degrees of Freedom<sup>6,9,10,13,94,111,154,242,368</sup>

Let  $F$  = degree of freedom of mechanism

$l$  = total number of links, including fixed link

$j$  = total number of joints

$f_i$  = degree of freedom of relative motion between element pairs of  $i$ th joint

Then, *in general*,

$$F = \lambda(l - j - 1) + \sum_{i=1}^j f_i \quad (3.1)$$

where  $\lambda$  is an integer whose value is determined as follows:

$\lambda = 3$ : Plane mechanisms with turning pairs, or turning and sliding pairs; spatial mechanisms with turning pairs only (motion on sphere); spatial mechanisms with rectangular sliding pairs only.

$\lambda = 6$ : Spatial mechanisms with lower pairs, the axes of which are nonparallel and nonintersecting; note exceptions such as listed under  $\lambda = 2$  and  $\lambda = 3$ . (See also Ref. 10.)

$\lambda = 2$ : Plane mechanisms with sliding pairs only; spatial mechanisms with “curved” sliding pairs only (motion on a sphere); three-link coaxial screw mechanisms.

Although included under Eq. (3.1), the motions on a sphere are usually referred to as special cases. For a comprehensive discussion and formulas including screw chains and other combinations of elements, see Ref. 13. The freedom of a mechanism with higher pairs should be determined from an equivalent lower-pair mechanism whenever feasible (see Sec. 3.2).

***Mechanism Characteristics Depending on Degree of Freedom Only.*** For plane mechanisms with turning pairs only and one degree of freedom,

$$2j - 3l + 4 = 0 \quad (3.2)$$

except in special cases. Furthermore, if this equation is valid, then the following are true:

1. The number of links is even.
2. The minimum number of binary links is four.
3. The maximum number of joints in a single link cannot exceed one-half the number of links.
4. If one joint connects  $m$  links, the joint is counted as  $(m - 1)$ -fold.

In addition, for nondegenerate plane mechanisms with turning and sliding pairs and one degree of freedom, the following are true:

1. If a link has only sliding elements, they cannot all be parallel.
2. Except for the three-link chain, binary links having sliding pairs only cannot, in general, be directly connected.

### 3. No closed nonrigid loop can contain less than two turning pairs.

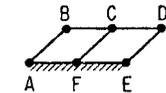
For plane mechanisms, having any combination of higher and/or lower pairs, and with one degree of freedom, the following hold:

1. The number of links may be odd.
2. The maximum number of elements in a link may exceed one-half the number of links, but an upper bound can be determined.<sup>154,368</sup>
3. If a link has only higher-pair connections, it must possess at least three elements.

For constrained spatial mechanisms in which Eq. (3.1) applies with  $\lambda = 6$ , the sum of the degrees of freedom of all joints must add up to 7 whenever the number of links is equal to the number of joints.

**Special Cases.**  $F$  can exceed the value predicted by Eq. (3.1) in certain special cases. These occur, generally, when a sufficient number of links are parallel in plane motion

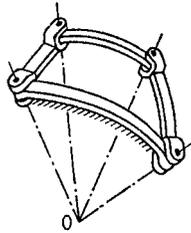
(Fig. 3.4a) or, in spatial motions, when the axes of the joints intersect (Fig. 3.4b—motion on a sphere, considered special in the sense that  $\lambda \neq 6$ ).



$$AB = CF = DE$$

$$AF = BC, FE = CD$$

(a)



(b)

**FIG. 3.4** Special cases that are exceptions to Eq. (3.1). (a) Parallelogram motion,  $F = 1$ . (b) Spherical four-bar mechanism,  $F = 1$ ; axes of four turning joints intersect at  $O$ .

The existence of these special cases or “critical forms” can sometimes also be detected by multigeneration effects involving pantographs, inversors, or mechanisms derived from these (see Sec. 3.6 and Ref. 154). In the general case, the critical form is associated with the singularity of the functional matrix of the differential displacement equations of the coordinates;<sup>130</sup> this singularity is usually difficult to ascertain, however, especially when higher pairs are involved. Known cases are summarized in Ref. 154. For

two-degree-of-freedom systems, additional results are listed in Refs. 111 and 242.

### 3.2.3 Creation of Mechanisms According to the Separation of Kinematic Structure and Function<sup>54,74,110,132,133</sup>

Basically this is an unbiased procedure for creating mechanisms according to the following sequence of steps:

1. Determine the basic characteristics of the desired motion (degree of freedom, plane or spatial) and of the mechanism (number of moving links, number of independent loops).
2. Find the corresponding kinematic chains from tables, such as in Ref. 133.
3. Find corresponding mechanisms by selecting joint types and fixed link in as many inequivalent ways as possible and sketch each mechanism.
4. Determine functional requirements and, if possible, their relationship to kinematic structure.
5. Eliminate mechanisms which do not meet functional requirements. Consider remaining mechanisms in greater detail and evaluate for potential use.

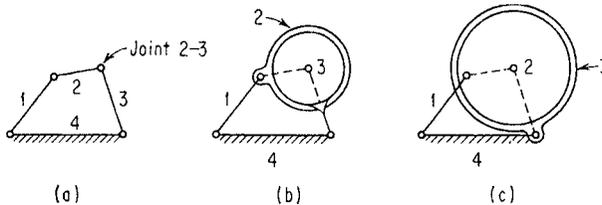
The method is described in greater detail in Refs. 110 and 133, which show applications to casement window linkages, constant-velocity shaft couplings, other mechanisms, and patent evaluation.

### 3.2.4 Kinematic Inversion

Kinematic inversion refers to the process of considering different links as the frame in a given kinematic chain. Thereby different and possibly useful mechanisms can be obtained. The slider crank, the turning-block and the swinging-block mechanisms are mutual inversions, as are also drag-link and “crank-and-rocker” mechanisms.

### 3.2.5 Pin Enlargement

Another method for developing different mechanisms from a base configuration involves enlarging the joints, illustrated in Fig. 3.5.



**FIG. 3.5** Pin enlargement. (a) Base configuration. (b) Enlarged pin at joint 2-3; pin part of link 3. (c) Enlarged pin at joint 2-3; pin takes place of link 2.

### 3.2.6 Mechanical Advantage

Neglecting friction and dynamic effects, the instantaneous power input and output of a mechanism must be equal and, in the absence of branching (one input, one output, connected by a single “path”), equal to the “power flow” through any other point of the mechanism.

In a single-degree-of-freedom mechanism without branches, the power flow at any point  $J$  is the product of the force  $F_j$  at  $J$ , and the velocity  $V_j$  at  $J$  in the direction of the force. Hence, for any point in such a mechanism,

$$F_j V_j = \text{constant} \quad (3.3)$$

neglecting friction and dynamic effects. For the point of input  $P$  and the point of output  $Q$  of such a mechanism, the mechanical advantage is defined as

$$MA = F_Q / F_P \quad (3.4)$$

### 3.2.7 Velocity Ratio

The “linear velocity ratio” for the motion of two points  $P$  and  $Q$  representing the input and output members or “terminals” of a mechanism is defined as  $V_Q / V_P$ . If input and

output terminals or links  $P$  and  $Q$  rotate, the “angular velocity ratio” is defined as  $\omega_Q/\omega_P$ , where  $\omega$  designates the angular velocity of the link. If  $T_Q$  and  $T_P$  refer to torque output and input in single-branch rotary mechanisms, the power-flow equation, in the absence of friction, becomes

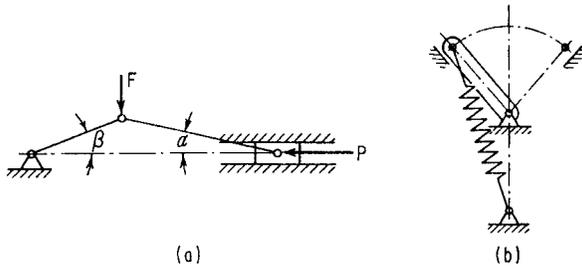
$$T_P \omega_P = T_Q \omega_Q \tag{3.5}$$

**3.2.8 Conservation of Energy**

Neglecting friction and dynamic effects, the product of the mechanical advantage and the linear velocity ratio is unity for all points in a single-degree-of-freedom mechanism without branch points, since  $F_Q V_Q / F_P V_P = 1$ .

**3.2.9 Toggle**

Toggle mechanisms are characterized by sudden snap or overcenter action, such as in Fig. 3.6*a* and *b*, schematics of a crushing mechanism and a light switch. The mechanical advantage, as in Fig. 3.6*a*, can become very high. Hence toggles are often used in such operations as clamping, crushing, and coining.

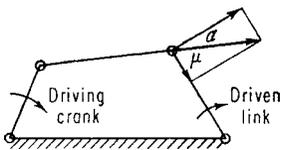


**FIG. 3.6** Toggle actions. (a)  $P/F = (\tan \alpha + \tan \beta)^{-1}$  (neglecting friction). (b) Schematic of a light switch.

**3.2.10 Transmission Angle**<sup>15,159,160,166–168,176,205</sup> (see Secs. 3.6 and 3.9)

The transmission angle  $\mu$  is used as a geometrical indication of the ease of motion of a mechanism under static conditions, excluding friction. It is defined by the ratio

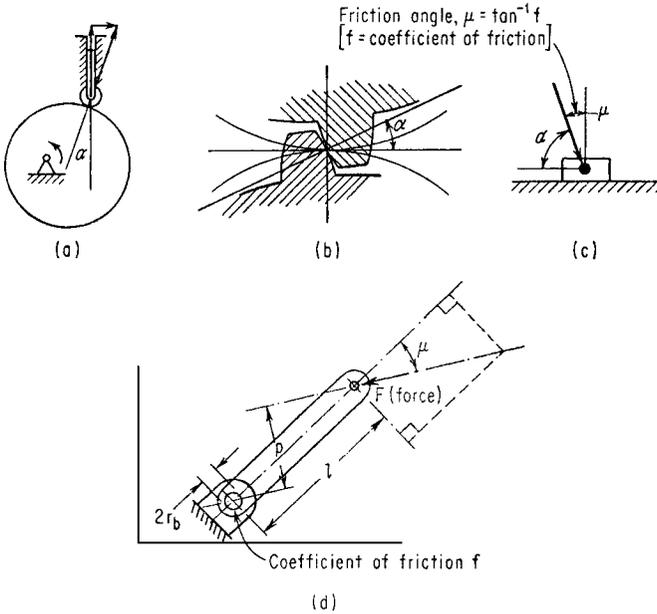
$$\tan \mu = \frac{\text{force component tending to move driven link}}{\text{force component tending to apply pressure on driven-link bearing or guide}} \tag{3.6}$$



**FIG. 3.7** Transmission angle  $\mu$  and pressure angle  $\alpha$  (also called the deviation angle) in a four-link mechanism.

where  $\mu$  is the transmission angle.

In four-link mechanisms,  $\mu$  is the angle between the coupler and the driven link (or the supplement of this angle) (Fig. 3.7) and has been used in optimizing linkage proportions (Secs. 3.6 and 3.9). Its ideal value is  $90^\circ$ ; in practice it may deviate from this value by  $30^\circ$  and possibly more.



**FIG. 3.8** Pressure angle (a) Cam and follower. (b) Gear teeth in mesh. (c) Link in sliding motion; condition of locking by friction  $(\alpha + \mu) \geq 90^\circ$ . (d) Conditions for locking by friction of a rotating link:  $\sin \mu \leq f r_b / l$ .

**3.2.11 Pressure Angle**

In cam and gear systems, it is customary to refer to the complement of the transmission angle, called the pressure angle  $\alpha$ , defined by the ratio

$$\tan \alpha = \frac{\text{force component tending to put pressure on follower bearing or guide}}{\text{force component tending to move follower}} \quad (3.7)$$

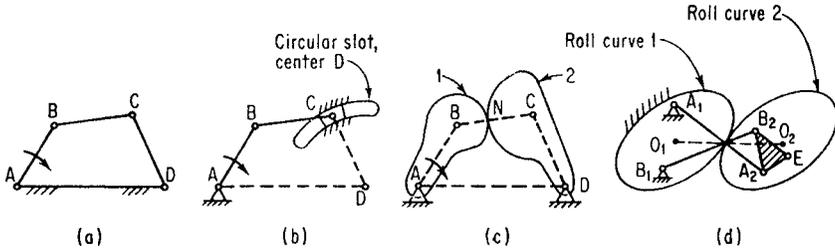
The ideal value of the pressure angle is zero; in practice it is frequently held to within  $30^\circ$  (Fig. 3.8). To ensure movability of the output member the ultimate criterion is to preserve a sufficiently large value of the ratio of driving force (or torque) to friction force (or torque) on the driven link. For a link in pure sliding (Fig. 3.8c), the motion will lock if the pressure angle and the friction angle add up to or exceed  $90^\circ$ .

A mechanism, the output link of which is shown in Fig. 3.8d, will lock if the ratio of  $p$ , the distance of the line of action of the force  $F$  from the fixed pivot axis, to the bearing radius  $r_b$  is less than or equal to the coefficient of friction,  $f$ , i.e., if the line of action of the force  $F$  cuts the "friction circle" of radius  $f r_b$ , concentric with the bearing.<sup>171</sup>

**3.2.12 Kinematic Equivalence**<sup>159,182,288,290,347,376</sup> (see Sec. 3.6)

"Kinematic equivalence," when applied to two mechanisms, refers to equivalence in motion, the precise nature of which must be defined in each case.

The motion of joint  $C$  in Fig. 3.9a and b is entirely equivalent if the quadrilaterals  $ABCD$  are identical; the motion of  $C$  as a function of the rotation of link  $AB$  is also



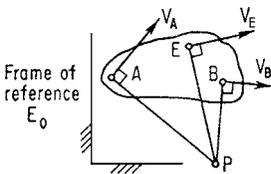
**FIG. 3.9** Kinematic equivalence: (a), (b), (c) for four-bar motion; (d) illustrates rolling motion and an equivalent mechanism. When  $O_1$  and  $O_2$  are fixed, curves are in rolling contact; when roll curve 1 is fixed and rolling contact is maintained,  $O_2$  generates circle with center  $O_1$ .

equivalent throughout the range allowed by the slot. In Fig. 3.9c,  $B$  and  $C$  are the centers of curvature of the contacting surfaces at  $N$ ;  $ABCD$  is one equivalent four-bar mechanism in the sense that, if  $AB$  is integral with body 1, the angular velocity and angular acceleration of link  $CD$  and body 2 are the same in the position shown, but not necessarily elsewhere.

Equivalence is used in design to obtain alternate mechanisms, which may be mechanically more desirable than the original. If, as in Fig. 3.9d,  $A_1A_2$  and  $B_1B_2$  are conjugate point pairs (see Sec. 3.4), with  $A_1B_1$  fixed on roll curve 1, which is in rolling contact with roll curve 2 ( $A_2B_2$  are fixed on roll curve 2), then the path of  $E$  on link  $A_2B_2$  and of the coincident point on the body of roll curve 2 will have the same path tangent and path curvature in the position shown, but not generally elsewhere.

**3.2.13 The Instant Center**

At any instant in the plane motion of a link, the velocities of all points on the link are proportional to their distance from a particular point  $P$ , called the instant center. The velocity of each point is perpendicular to the line joining that point to  $P$  (Fig. 3.10).



**FIG. 3.10** Instant center,  $P$ .  $V_E/V_B = EP/BP$ ,  $V_E \perp EP$ , etc.

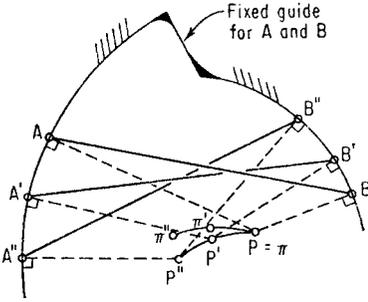
Regarded as a point on the link,  $P$  has an instantaneous velocity of zero. In pure rectilinear translation,  $P$  is at infinity.

The instant center is defined in terms of velocities and is not the center of path curvature for the points on the moving link in the instant shown, except in special cases, e.g., points on common tangent between centrodes (see Sec. 3.4).

An extension of this concept to the “instantaneous screw axis” in spatial motions has been described.<sup>38</sup>

**3.2.14 Centroides, Polodes, Pole Curves**

Relative plane motion of two links can be obtained from the pure rolling of two curves, the “fixed” and “movable centrodes” (“polodes” and “pole curves,” respectively),



**FIG. 3.11** Construction of fixed and movable centroids. Link  $AB$  in plane motion, guided at both ends;  $PP' = P\pi'$ ;  $\pi'\pi'' = P'P''$ , etc.

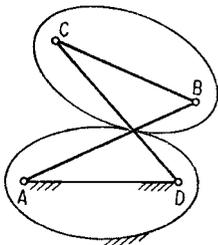
which can be constructed as illustrated in the following example.

As shown in Fig. 3.11, the intersections of path normals locate successive instant centers  $P, P', P'', \dots$ , whose locus constitutes the fixed centrode. The movable centrode can be obtained either by inversion (i.e., keeping  $AB$  fixed, moving the guide, and constructing the centrode as before) or by "direct construction": superposing triangles  $A'B'P', A''B''P'', \dots$ , on  $AB$  so that  $A'$  covers  $A$  and  $B'$  covers  $B$ , etc. The new locations thus found for  $P', P'', \dots$ , marked  $\pi', \pi'', \dots$ , then constitute points on the movable centrode, which rolls without slip on the fixed centrode and carries  $AB$  with it.

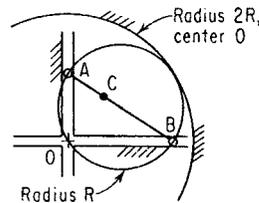
it, duplicating the original motion. Thus, for the motion of  $AB$ , the centrode-rolling motion is kinematically equivalent to the original guided motion.

In the antiparallel equal-crank linkage, with the shortest link fixed, the centrodes for the coupler motion are identical ellipses with foci at the link pivots (Fig. 3.12); if the longer link  $AB$  were held fixed, the centrodes for the coupler motion of  $CD$  would be identical hyperbolas with foci at  $A, B$ , and  $C, D$ , respectively.

In the elliptic trammel motion (Fig. 3.13) the centrodes are two circles, the smaller rolling inside the larger, twice its size. Known as "cardanic motion," it is used in press drives, resolvers, and straight-line guidance.



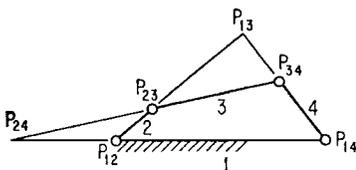
**FIG. 3.12** Antiparallel equal-crank linkage; rolling ellipses, foci at  $A, D, B, C$ ;  $AD < AB$ .



**FIG. 3.13** Cardanic motion of the elliptic trammel, so called because any point  $C$  of  $AB$  describes an ellipse; midpoint of  $AB$  describes circle, center  $O$  (point  $C$  need not be collinear with  $AB$ ).

Apart from their use in kinematic analysis, the centrodes are used to obtain alternate, kinematically equivalent mechanisms, and sometimes to guide the original mechanism past the "in-line" or "dead-center" positions.<sup>207</sup>

**3.2.15 The Theorem of Three Centers**



**FIG. 3.14** Instant centers in four-bar motion.

Also known as Kennedy's or the Aronhold-Kennedy theorem, this theorem states that, for any three bodies  $i, j, k$  in plane motion, the relative instant centers  $P_{ij}, P_{jk}, P_{ki}$  are collinear; here  $P_{ij}$ , for instance, refers to the instant center of the motion of link  $i$  relative to link  $j$ , or vice versa. Figure 3.14 illustrates the theorem

with respect to four-bar motion. It is used in determining the location of instant centers and in planar path curvature investigations.

### 3.2.16 Function, Path, and Motion Generation

In “function generation” the input and output motions of a mechanism are linear analogs of the variables of a function  $F(x, y, \dots) = 0$ . The number of degrees of freedom of the mechanism is equal to the number of independent variables.

For example, let  $\phi$  and  $\psi$ , the linear or rotary motions of the input and output links or “terminals,” be linear analogs of  $x$  and  $y$ , where  $y = f(x)$  within the range  $x_0 \leq x \leq x_{n+1}$ ,  $y_0 \leq y \leq y_{n+1}$ . Let the *input* values  $\phi_0, \phi_j, \phi_{n+1}$  and the *output* values  $\psi_0, \psi_j, \psi_{n+1}$  correspond to the values  $x_0, x_j, x_{n+1}$  and  $y_0, y_j, y_{n+1}$ , of  $x$  and  $y$ , respectively, where the subscripts 0,  $j$ , and  $(n + 1)$  designate starting,  $j$ th intermediate, and terminal values. Scale factors  $r_\phi, r_\psi$  are defined by

$$r_\phi = (x_{n+1} - x_0)/(\phi_{n+1} - \phi_0) \quad r_\psi = (y_{n+1} - y_0)/(\psi_{n+1} - \psi_0)$$

(it is assumed that  $y_0 \neq y_{n+1}$ ), such that  $y - y_i = r_\psi(\psi - \psi_j)$ ,  $x - x_j = r_\phi(\phi - \phi_j)$ , whence

$$d\psi/d\phi = (r_\psi/r_\phi)(dy/dx), \quad d^2\psi/d\phi^2 = (r_\psi^2/r_\phi)(d^2y/dx^2), \quad \text{and generally,}$$

$$d^n\psi/d\phi^n = (r_\psi^n/r_\phi)(d^n y/dx^n)$$

In “path generation” a point of a floating link traces a prescribed path with reference to the frame. In “motion generation” a mechanism is designed to conduct a floating link through a prescribed sequence of positions (Ref. 382). Positions along the path or specification of the prescribed motion may or may not be coordinated with input displacements.

### 3.3 PRELIMINARY DESIGN ANALYSIS: DISPLACEMENTS, VELOCITIES, AND

**ACCELERATIONS** (Refs. 41, 58, 61, 62, 96, 116, 117, 129, 145, 172, 181, 194, 212, 263, 278, 298, 302, 309, 361, 384, 428, 487; see also Sec. 3.9)

Displacements in mechanisms are obtained graphically (from scale drawings) or analytically or both. Velocities and accelerations can be conveniently analyzed graphically by the “vector-polygon” method or analytically (in case of plane motion) via complex numbers. In all cases, the “vector equation of closure” is utilized, expressing the fact that the mechanism forms a closed kinematic chain.

#### 3.3.1 Velocity Analysis: Vector-Polygon Method

The method is illustrated using a point  $D$  on the connecting rod of a slider-crank mechanism (Fig. 3.15). The vector-velocity equation for  $C$  is

$$\mathbf{V}_C = \mathbf{V}_B + \mathbf{V}_{CB}^n + \mathbf{V}_{CB}^t = \text{a vector parallel to line AX}$$

where  $\mathbf{V}_C$  = velocity of  $C$  (Fig. 3.15)

$\mathbf{V}_B$  = velocity of  $B$

$V_{C/B}^n$  = normal component of velocity of  $C$  relative to  $B$  = component of relative velocity along  $BC$  = zero (owing to the rigidity of the connecting rod)  
 $V_{C/B}^t$  = tangential component of velocity of  $C$  relative to  $B$ , value  $(\omega_{BC}) BC$ , perpendicular to  $BC$

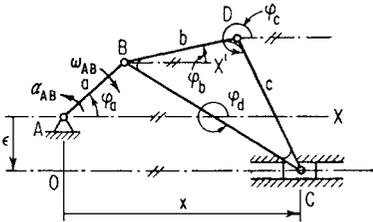


FIG. 3.15 Offset slider-crank mechanism.

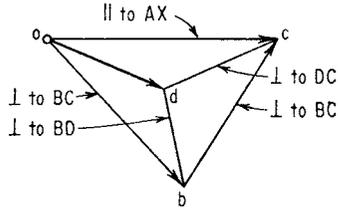


FIG. 3.16 Velocity polygon for slider-crank mechanism of Fig. 3.15.

The velocity equation is now “drawn” by means of a vector polygon as follows:

1. Choose an arbitrary origin  $o$  (Fig. 3.16).
2. Label terminals of velocity vectors with lowercase letters, such that absolute velocities start at  $o$  and terminate with the letter corresponding to the point whose velocity is designated. Thus  $V_B = ob$ ,  $V_C = oc$ , to a certain scale.
3. Draw  $ob = (\omega_{AB}) AB/k_v$ , where  $k_v$  is the velocity scale factor, say, inches per inch per second.
4. Draw  $bc \perp BC$  and  $oc \parallel AX$  to determine intersection  $c$ .
5. Then  $V_C = (oc)/k_v$ ; absolute velocities always start at  $o$ .
6. Relative velocities  $V_{C/B}$ , etc., connect the terminals of absolute velocities. Thus  $V_{C/B} = (bc)/k_v$ . Note the reversal of order in  $C/B$  and  $bc$ .
7. To determine the velocity of  $D$ , one way is to write the appropriate velocity-vector equation and draw it on the polygon:  $V_D = V_C + V_{D/C}^n + V_{D/C}^t$ ; the second is to utilize the “principle of the velocity image.” This principle states that  $\Delta bcd$  in the velocity polygon is similar to  $\Delta BCD$  in the mechanism, and the sense  $b \rightarrow c \rightarrow d$  is the same as that of  $B \rightarrow C \rightarrow D$ . This “image construction” applies to any three points on a rigid link in plane motion. It has been used in Fig. 3.16 to locate  $d$ , whence  $V_D = (od)/k_v$ .
8. The angular velocity  $\omega_{BC}$  of the coupler can now be determined from

$$|\omega_{BC}| = \frac{|V_{B/C}|}{BC} = \frac{|(cb)/k_v|}{BC}$$

The sense of  $\omega_{BC}$  is determined by imagining  $B$  fixed and observing the sense of  $V_{C/B}$ . Here  $\omega_{BC}$  is counterclockwise.

9. Note that to determine the velocity of  $D$  it is easier to proceed in steps, to determine the velocity of  $C$  first and thereafter to use the image-construction method.

### 3.3.2 Velocity Analysis: Complex-Number Method

Using the slider crank of Fig. 3.15 once more as an illustration with  $x$  axis along the center line of the guide, and recalling that  $i^2 = -1$ , we write the complex-number equations as follows, with the equivalent vector equation below each:

Displacement:  $ae^{i\varphi_a} + be^{i\varphi_b} + ce^{i\varphi_c} = x$  (3.8)

$$\mathbf{AB} + \mathbf{BD} + \mathbf{DC} = \mathbf{AC}$$

Velocity:  $iae^{i\varphi_a}\omega_{AB} + (ibe^{i\varphi_b} + ice^{i\varphi_c})\omega_{BC} = dx/dt$  ( $t = \text{time}$ ) (3.9)

$$\mathbf{V}_B + \mathbf{V}_{D/B} + \mathbf{V}_{C/D} = \mathbf{V}_C$$

Note that  $\omega_{AB} = d\varphi_a/dt$  is positive when counterclockwise and negative when clockwise; in this problem  $\omega_{AB}$  is negative.

The complex conjugate of Eq. (3.9)

$$-iae^{-i\varphi_a}\omega_{AB} - (ibe^{-i\varphi_b} + ice^{-i\varphi_c})\omega_{BC} = dx/dt$$
 (3.10)

From Eqs. (3.9) and (3.10), regarded as simultaneous equations:

$$\frac{\omega_{BC}}{\omega_{AB}} = \frac{ia(e^{i\varphi_a} + e^{-i\varphi_a})}{-ib(e^{i\varphi_b} + e^{-i\varphi_b}) - ic(e^{i\varphi_c} + e^{-i\varphi_c})} = \frac{a \cos \varphi_a}{-(b \cos \varphi_b + c \cos \varphi_c)}$$

$$\mathbf{V}_D = \mathbf{V}_B + \mathbf{V}_{D/B} = ia e^{i\varphi_a} \omega_{AB} + ib e^{i\varphi_b} \omega_{BC}$$

The quantities  $\varphi_a, \varphi_b, \varphi_c$  are obtained from a scale drawing or by trigonometry.

Both the vector-polygon and the complex-number methods can be readily extended to accelerations, and the latter also to the higher accelerations.

### 3.3.3 Acceleration Analysis: Vector-Polygon Method

We continue with the slider crank of Fig. 3.15. After solving for the velocities via the velocity polygon, write out and “draw” the acceleration equations. Again proceed in order of increasing difficulty: from  $B$  to  $C$  to  $D$ , and determine first the acceleration of point  $C$ :

$$\mathbf{A}_C = \mathbf{A}_C^n = \mathbf{A}_C^t = \mathbf{A}_B^n + \mathbf{A}_B^t + \mathbf{A}_{C/B}^n + \mathbf{A}_{C/B}^t$$

where  $\mathbf{A}_C^n$  = acceleration normal to path of  $C$  (equal to zero in this case)

$\mathbf{A}_C^t$  = acceleration parallel to path of  $C$

$\mathbf{A}_B^n$  = acceleration normal to path of  $B$ , value  $\omega_{AB}^2(\overline{AB})$ , direction  $B$  to  $A$

$\mathbf{A}_B^t$  = acceleration parallel to path of  $B$ , value  $\alpha_{AB}(AB)$ ,  $\perp AB$ , sense determined by that of  $\alpha_{AB}$  (where  $\alpha_{AB} = d\omega_{AB}/dt$ )

$\mathbf{A}_{C/B}^n$  = acceleration component of  $C$  relative to  $B$ , in the direction  $C$  to  $B$ , value  $(BC)\omega_{BC}^2$

$\mathbf{A}_{C/B}^t$  = acceleration component of  $C$  relative to  $B$ ,  $\perp \overline{BC}$ , value  $\alpha_{BC}(\overline{BC})$ . Since  $\alpha_{BC}$  is unknown, so is the magnitude and sense of  $\mathbf{A}_{C/B}^t$

The acceleration polygon is now drawn as follows (Fig. 3.17):

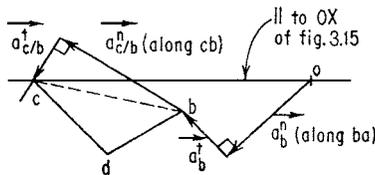


FIG. 3.17 Acceleration polygon for slider crank of Fig. 3.15.  $\Delta bcd \approx \Delta BCD$  of Fig. 3.15.

1. Choose an arbitrary origin  $o$ , as before.
2. Draw each acceleration of scale  $k_a$  (inch per inch per second squared), and label the appropriate vector terminals with the lowercase letter corresponding to the point whose acceleration is designated, e.g.,  $\mathbf{A}_B = (\mathbf{ob})/k_a$ . Draw  $\mathbf{A}_B^n, \mathbf{A}_B^t$  and  $\mathbf{A}_{C/B}^n$ .

3. Knowing the direction of  $\mathbf{A}_{C/B}^t (\perp \overline{BC})$ , and also of  $\mathbf{A}_C$  (along the slide), locate  $c$  at the intersection of a line through  $o$ , parallel to  $AX$ , and the line representing  $\mathbf{A}_{C/B}^t$ .  $\mathbf{A}_C = (\mathbf{oc})/k_a$ .
4. The acceleration of  $D$  is obtained using the “principle of the acceleration image,” which states that, for any three points on a rigid body, such as link  $BCD$ , in plane motion,  $\Delta bcd$  and  $\Delta BCD$  are similar, and the sense  $b \rightarrow c \rightarrow d$  is the same as that of  $B \rightarrow C \rightarrow D$ .  $\mathbf{A}_D = (\mathbf{od})/k_a$ .
5. Relative accelerations can also be found from the polygon. For instance,  $\mathbf{A}_{C/D} = (\mathbf{dc})/k_a$ ; note reversal of order of the letters  $C$  and  $D$ .
6. The angular acceleration  $\alpha_{BC}$  of the connecting rod can now be determined from  $|\alpha_{BC}| = |\mathbf{A}_{C/B}^t|/BC$ . Its sense is determined by that of  $\mathbf{A}_{C/B}^t$ .
7. The acceleration of  $D$  can also be obtained by direct drawing of the equation  $\mathbf{A}_D = \mathbf{A}_C + \mathbf{A}_{D/C}$ .

**3.3.4 Acceleration Analysis: Complex-Number Method** (see Fig. 3.15)

Differentiating Eq. (3.9), obtain the acceleration equation of the slider-crank mechanism:

$$ae^{i\varphi_a}(i\alpha_{AB} - \omega_{AB}^2) + (be^{i\varphi_b} + ce^{i\varphi_c})(i\alpha_{BC} - \omega_{BC}^2) = d^2x/dt^2 \tag{3.11}$$

This is equivalent to the vector equation

$$\mathbf{A}_B^t + \mathbf{A}_B^n + \mathbf{A}_{D/B}^t + \mathbf{A}_{D/B}^n + \mathbf{A}_{C/D}^t + \mathbf{A}_{C/D}^n = \mathbf{A}_C^n + \mathbf{A}_C^t$$

Combining Eq. (3.11) and its complex conjugate, eliminate  $d^2x/dt^2$  and solve for  $\alpha_{BC}$ . Substitute the value of  $\alpha_{BC}$  in the following equation for  $\mathbf{A}_D$ :

$$\mathbf{A}_D + \mathbf{A}_B + \mathbf{A}_{D/B} = ae^{i\varphi_a}(i\alpha_{AB} - \omega_{AB}^2) + be^{i\varphi_b}(i\alpha_{BC} - \omega_{BC}^2)$$

The above complex-number approach also lends itself to the analysis of motions involving Coriolis acceleration. The latter is encountered in the determination of the relative acceleration of two instantaneously coincident points on different links.<sup>106,171,384</sup> The general complex-number method is discussed more fully in Ref. 381. An alternate approach, using the acceleration center, is described in Sec. 3.4. The accelerations in certain specific mechanisms are discussed in Sec. 3.9.

**3.3.5 Higher Accelerations** (see also Sec. 3.4)

The second acceleration (time derivative of acceleration), also known as “shock,” “jerk,” or “pulse,” is significant in the design of high-speed mechanisms and has been investigated in several ways.<sup>41,61,62,106,298,381,384,487</sup> It can be determined by direct differentiation of the complex-number acceleration equation.<sup>381</sup> The following are the basic equations:

*Shock of B Relative to A (where A and B represent two points on one link whose angular velocity is  $\omega_p$ ;  $\alpha_p = d\omega_p/dt$ ).*<sup>298</sup> Component along AB:

$$-3\alpha_p\omega_p AB$$

Component perpendicular to AB:

$$AB(d\alpha_p/dt - \omega_p^3)$$

in direction of  $\omega_p \times \mathbf{AB}$ .

*Absolute Shock*.<sup>298</sup> Component along path tangent (in direction of  $\omega_p \times \mathbf{AB}$ ):

$$d^2v/dt^2 - v^3/\rho^2$$

where  $v$  = velocity of  $B$  and  $\rho$  = radius of curvature of path of  $B$ .

Component directed toward the center of curvature:

$$\frac{v}{\rho} \left[ 3 \frac{dv}{dt} - \frac{v}{\rho} \frac{d\rho}{dt} \right]$$

*Absolute Shock with Reference to Rolling Centroides* (Fig. 3.18, Sec. 3.4) [ $l, m$  as in Eq. (3.22)]. Component along  $AP$ :

$$-3\omega_p^3 \delta \left[ \delta \sin \psi \left( \frac{1}{m} + \frac{1}{g} \right) + \delta \cos \psi \left( \frac{1}{l} - \frac{1}{\delta} \right) - \frac{r}{g} \right] \quad g = \frac{\omega_p^2 \delta}{\alpha_p}$$

Component perpendicular to  $AP$  in direction of  $\omega_p \times \mathbf{PA}$ :

$$r \left( \frac{d\alpha_p}{dt} - \omega_p^3 \right) + 3\delta^2 \omega_p^3 \left[ \cos \psi \left( \frac{1}{m} + \frac{1}{g} \right) - \sin \psi \left( \frac{1}{l} - \frac{1}{g} \right) \right]$$

### 3.3.6 Accelerations in Complex Mechanisms

When the number of real unknowns in the complex-number or vector equations is greater than two, several methods can be used.<sup>106,145,309</sup> These are applicable to mechanisms with more than four links.

### 3.3.7 Finite Differences in Velocity and Acceleration Analysis<sup>212,375,419,428</sup>

When the time-displacement curve of a point in a mechanism is known, the calculus of finite differences can be used for the calculation of velocities and accelerations. The data can be numerical or analytical. The method is useful also in ascertaining the existence of local fluctuations in velocities and accelerations, such as occur in cam-follower systems, for instance.

Let a time-displacement curve be subdivided into equal time intervals  $\Delta t$  and define the  $i$ th, the general interval, as  $t_i \leq t \leq t_{i+1}$ , such that  $\Delta t = t_{i+1} - t_i$ . The “central-difference” formulas then give the following approximate values for velocities  $dy/dt$ , accelerations  $d^2y/dt^2$ , and shock  $d^3y/dt^3$ , where  $y_i$  denotes the displacement  $y$  at the time  $t = t_i$ :

$$\text{Velocity at } t = t_i + \frac{1}{2}\Delta t: \quad \frac{dy}{dt} = \frac{y_{i+1} - y_i}{\Delta t} \tag{3.12}$$

$$\text{Acceleration at } t = t_i: \quad \frac{d^2y}{dt^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta t)^2} \tag{3.13}$$

$$\text{Shock at } t = t_i + \frac{1}{2}\Delta t: \quad \frac{d^3y}{dt^3} = \frac{y_{i+2} - 3y_{i+1} + 3y_i - y_{i-1}}{(\Delta t)^3} \tag{3.14}$$

If the values of the displacements  $y_i$  are known with absolute precision (no error), the values for velocities, accelerations, and shock in the above equations become increasingly accurate as  $\Delta t$  approaches zero, provided the curve is smooth. If, however, the displacements  $y_i$  are known only within a given tolerance, say  $\pm \epsilon y$ , then the accuracy of the computations will be high only if the interval  $\Delta t$  is sufficiently small and, in addition, if

$$\begin{aligned} 2\epsilon y/\Delta t &<< dy/dt && \text{for velocities} \\ 4\epsilon y/(\Delta t)^2 &<< d^2y/dt^2 && \text{for accelerations} \\ 8\epsilon y/(\Delta t)^3 &<< d^3y/dt^3 && \text{for shock} \end{aligned}$$

and provided also that these requirements are mutually compatible.

Further estimates of errors resulting from the use of Eqs. (3.12), (3.13), and (3.14), as well as alternate formulations involving “forward” and “backward” differences, are found in texts on numerical mathematics (e.g., Ref. 193, pp. 94–97 and 110–112, with a discussion of truncation and round-off errors).

The above equations are particularly useful when the displacement-time curve is given in the form of a numerical table, as frequently happens in checking an existing design and in redesigning.

Some current computer programs in displacement, velocity, and acceleration analysis are listed in Ref. 129; the kinematic properties of specific mechanisms, including spatial mechanisms, are summarized in Sec. 3.9.<sup>129</sup>

### 3.4 PRELIMINARY DESIGN ANALYSIS: PATH CURVATURE

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The following principles apply to the analysis of a mechanism in a given position, as well as to synthesis when motion characteristics are prescribed in the vicinity of a particular position. The technique can be used to obtain a quick “first approximation” to mechanism proportions which can be refined at a later stage.

#### 3.4.1 Polar-Coordinate Convention

Angles are measured counterclockwise from a directed line segment, the “pole tangent”  $PT$ , origin at  $P$  (see Fig. 3.18); the polar coordinates  $(r, \psi)$  of a point  $A$  are either  $r = |PA|$ ,  $\psi = \angle TPA$  or  $r = -|PA|$ ,  $\psi = \angle TPA \pm 180^\circ$ . For example, in Fig. 3.18  $r$  is positive, but  $r_c$  is negative.

#### 3.4.2 The Euler-Savary Equation (Fig. 3.18)

$PT$  = common tangent of fixed and moving centrodes at point of contact  $P$  (the instant center).

$PN$  = principal normal at  $P$ ;  $\angle TPN = 90^\circ$ .

$PA$  = line or ray through  $P$ .

$C_A(r_c, \psi)$  = center of curvature of path of  $A(r, \psi)$  in position shown.  $A$  and  $C_A$  are called “conjugate points.”

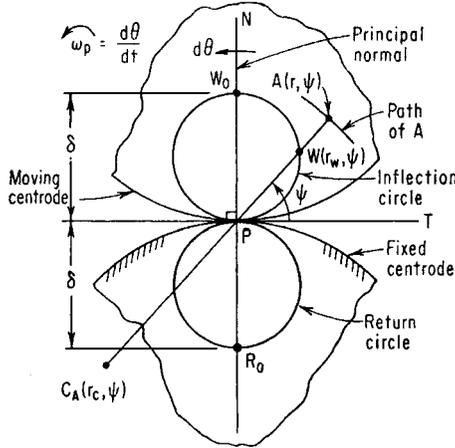


FIG. 3.18 Notation for the Euler-Savary equation.

$\theta$  = angle of rotation of moving centrode, positive counterclockwise.

$s$  = arc length along fixed centrode, measured from  $P$ , positive toward  $T$ .

The Euler-Savary equation is valid under the following assumptions:

1. During an infinitesimal displacement from the position shown,  $d\theta/ds$  is finite and different from zero.
2. Point  $A$  does not coincide with  $P$ .
3.  $AP$  is finite.

Under these conditions, the curvature of the path of  $A$  in the position shown can be determined from the following ‘‘Euler-Savary’’ equations:

$$[(1/r) - (1/r_c)] \sin \psi = -d\theta/ds = -\omega_p/v_p \tag{3.15}$$

where  $\omega_p$  = angular velocity of moving centrode

=  $d\theta/dt$ ,  $t$  = time

$v_p$  = corresponding velocity of point of contact between centrodes along the fixed centrode

=  $ds/dt$

Let  $r_w$  = polar coordinate of point  $W$  on ray  $PA$ , such that radius of curvature of path of  $W$  is infinite in the position shown; then  $W$  is called the ‘‘inflection point’’ on ray  $PA$ , and

$$1/r - 1/r_c = 1/r_w \tag{3.16}$$

The locus of all inflection points  $W$  in the moving centrode is the ‘‘inflection circle,’’ tangent to  $PT$  at  $P$ , of diameter  $PW_0 = \delta = -ds/d\theta$ , where  $W_0$ , the ‘‘inflection pole,’’ is the inflection point on the principal normal ray. Hence,

$$[(1/r) - (1/r_c)] \sin \psi = 1/\delta \tag{3.17}$$

The centers of path curvature of all points at infinity in the moving centrode are on the “return circle,” also of diameter  $\delta$ , and obtained as the reflection of the inflection circle about line  $PT$ . The reflection of  $W_0$  is known as the “return pole”  $R_0$ . For the pole velocity (the time rate change of the position of  $P$  along the fixed centrode as the motion progresses, also called the “pole transfer velocity”<sup>421f</sup>) we have

$$v_p = ds/dt = -\omega_p \delta \tag{3.18}$$

The curvatures of the paths of all points on a given ray are concave toward the inflection point on that ray.

For the diameter of the inflection and return circles we have

$$\delta = r_p r_\pi / (r_\pi - r_p) \tag{3.19}$$

where  $r_p$  and  $r_\pi$  are the polar coordinates of the centers of curvature of the moving and fixed centrodes, respectively, at  $P$ . Let  $\rho = r_c - r$  be the instantaneous value of the radius of curvature of the path of  $A$ , and  $w = AW$ , then

$$r^2 = \rho w \tag{3.20}$$

which is known as the “quadratic form” of the Euler-Savary equation.

Conjugate points in the planes of the moving and fixed centrodes are related by a “quadratic transformation.”<sup>32</sup> When the above assumptions 1, 2, and 3, establishing the validity of the Euler-Savary equations, are not satisfied, see Ref. 281; for a further curvature theorem, useful in relative motions, see Ref. 23. For a computer-compatible complex-number treatment of path curvature theory, see Ref. 421f, Chap. 4.

**EXAMPLE** Cylinder of radius 2 in, rolling inside a fixed cylinder of radius 3 in, common tangent horizontal, both cylinders above the tangent,  $\delta = 6$  in,  $W_0(6, 90^\circ)$ . For point  $A_1(\sqrt{2}, 45^\circ)$ ,  $r_{c1} = 1.5\sqrt{2}$ ,  $C_{A1}(1.5\sqrt{2}, 45^\circ)$ ,  $\rho_{A1} = 0.5\sqrt{2}$ ,  $r_{w1} = 3\sqrt{2}$ ,  $v_p = -6\omega_p$ . For point  $A_2(-\sqrt{2}, 135^\circ)$ ,  $r_{c2} = -0.75\sqrt{2}$ ,  $C_{A2}(-0.75\sqrt{2}, 135^\circ)$ ,  $\rho_{A2} = 0.25\sqrt{2}$ ,  $r_{w2} = 3\sqrt{2}$ .

*Complex-number forms* of the Euler-Savary equation<sup>393,421f</sup> and related expressions are independent of the choice of the  $x, iy$  coordinate system. They correlate the following complex vectors on any one ray (see Fig. 3.18):  $\mathbf{a} = \mathbf{PA}$ ,  $\mathbf{w} = \mathbf{PW}$ ,  $\mathbf{c} = \mathbf{PC}_A$  and  $\mathbf{p} = \mathbf{C}_A A$ , each expressed explicitly in terms of the others:

1. If points  $P, A$ , and  $W$  are known, find  $C_A$  by

$$\mathbf{p} = (a^2/\mathbf{a} - \mathbf{w}) e^{i \arg(\mathbf{a} - \mathbf{w})}$$

where  $a = |\mathbf{a}|$ .

2. If points  $P, A$ , and  $C_A$  are known, find  $W$  by

$$\mathbf{w} = \mathbf{a} - (a/\rho)^2 \mathbf{p}$$

where  $\rho = |\mathbf{p}|$ .

3. If points  $P, W$ , and  $C_A$  are known, find  $A$  by

$$\mathbf{a} = \mathbf{w}/(\mathbf{w} + \mathbf{c})$$

4. If points  $A, C_A$ , and  $W$  are known, find  $P$  by

$$\mathbf{a} = (|\mathbf{WA}| \rho)^{1/2} (\pm e^{i \arg \mathbf{p}})$$

Note that the last equation yields two possible locations for  $P$ , symmetric about  $A$ . This is borne out also by Bobillier's construction (see Ref. 421f, Fig. 4.29, p. 329).

5. The vector diameter of the inflection circle,  $\delta = \mathbf{PW}_0$ , in complex notation:

$$\delta = -\mathbf{r}_p \mathbf{r}_\pi / (\mathbf{r}_p - \mathbf{r}_\pi) \tag{3.19a}$$

where  $\mathbf{r}_p = \mathbf{O}_p \mathbf{P}$ ,  $\mathbf{r}_\pi = \mathbf{O}_\pi \mathbf{P}$  and  $O_p$  and  $O_\pi$  are the centers of curvature of the fixed and moving centrodes, respectively.

6. The pole velocity in complex vector form is

$$\mathbf{v}_p = i\omega_\pi \delta \tag{3.18a}$$

where  $\omega_\pi$  is the angular velocity of the moving centrode.

7. If points  $P$ ,  $A$ , and  $W_0$  are known:

$$\mathbf{w} = \cos(\arg \mathbf{a} - \arg \delta) \delta e^{i(\arg \mathbf{a} - \arg \delta)}$$

With the data of the above example, letting  $PT$  be the positive  $x$  axis and  $PN$  the positive  $iy$  axis, we have  $\mathbf{r}_p = -i2$ ,  $\mathbf{r}_\pi = -i3$ ;  $\delta = -(-i2)(-i3)/(-i3 + i2) = i6$ , which is the same as the vector locating the inflection pole  $W_0$ ,  $\mathbf{w}_0 = \mathbf{PW}_0 = i6$ . For point  $A_1$ ,

$$\begin{aligned} \mathbf{a}_1 &= \sqrt{2}e^{i45^\circ} & \mathbf{w}_1 &= \cos(45^\circ - 90^\circ)i6e^{i(45^\circ - 90^\circ)} = 3\sqrt{2}e^{i45^\circ} \\ \boldsymbol{\rho}_{A_1} &= (2/\sqrt{2}e^{i45^\circ} - 3\sqrt{2}e^{i45^\circ}) \exp[i \arg(\sqrt{2}e^{i45^\circ} - 3\sqrt{2}e^{i45^\circ})] = (\sqrt{2}/2)e^{i(-135^\circ)} \\ \mathbf{c}_{A_1} &= \mathbf{a}_1 - \boldsymbol{\rho}_{A_1} = \sqrt{2}e^{i45^\circ} - (\sqrt{2}/2)e^{i(-135^\circ)} = (3\sqrt{2}/2)e^{i45^\circ} \\ \mathbf{v}_p &= i\omega_\pi i6 = -\omega 6 \end{aligned}$$

For point  $A_2$ ,

$$\begin{aligned} \mathbf{a}_2 &= \sqrt{2}e^{i(-45^\circ)} & \mathbf{w}_2 &= \cos(-45^\circ - 90^\circ)i6e^{i(-45^\circ - 90^\circ)} = 3\sqrt{2}e^{i135^\circ} \\ \boldsymbol{\rho}_{A_2} &= (2/\sqrt{2}e^{i(-45^\circ)} - 3\sqrt{2}e^{i135^\circ}) \exp[i \arg(\sqrt{2}e^{i(-45^\circ)} - 3\sqrt{2}e^{i135^\circ})] = (\sqrt{2}/4)e^{i(-45^\circ)} \end{aligned}$$

and  $\mathbf{C}_{A_2} = \mathbf{a}_2 - \boldsymbol{\rho}_{A_2} = (\sqrt{2} - \sqrt{2}/4)e^{i(-45^\circ)} = (3\sqrt{2}/4)e^{i(-45^\circ)}$

Note that these are equal to the previous results and are readily programmed in a digital computer.

Graphical constructions paralleling the four forms of the Euler-Savary equation are given in Refs. 394 and 421f, p. 3.27.

### 3.4.3 Generating Curves and Envelopes<sup>368</sup>

Let  $g-g$  be a smooth curve attached to the moving centrode and  $e-e$  be the curve in the fixed centrode enveloping the successive positions of  $g-g$  during the rolling of the centrodes. Then  $g-g$  is called a "generating curve" and  $e-e$  its "envelope" (Fig. 3.19).

If  $C_g$  is the center of curvature of  $g-g$  and  $C_e$  that of  $e-e$  (at  $M$ ):

1.  $C_e$ ,  $P$ ,  $M$ , and  $C_g$  are collinear ( $M$  being the point of contact between  $g-g$  and  $e-e$ ).
2.  $C_e$  and  $C_g$  are conjugate points, i.e., if  $C_g$  is considered a point of the moving centrode,

the center of curvature of its path lies at  $C_e$ ; interchanging the fixed and moving centrodes will invert this relationship.

3. Aronhold's first theorem: The return circle is the locus of the centers of curvature of all envelopes whose generating curves are straight lines.
4. If a straight line in the moving plane always passes through a fixed point by sliding through it and rotating about it, that point is on the return circle.
5. Aronhold's second theorem: The inflection circle is the locus of the centers of curvature for all generating curves whose envelopes are straight lines.

**EXAMPLE** (utilizing 4 above): In the swinging-block mechanism of Fig. 3.20, point  $C$  is on the return circle, and the center of curvature of the path of  $C$  as a point of link  $BD$  is therefore at  $C_c$ , halfway between  $C$  and  $P$ . Thus  $ABCC_c$  constitutes a four-bar mechanism, with  $C_c$  as a fixed pivot, equivalent to the original mechanism in the position shown with reference to path tangents and path curvatures of points in the plane of link  $BD$ .

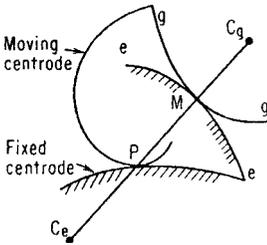


FIG. 3.19 Generating curve and envelope.

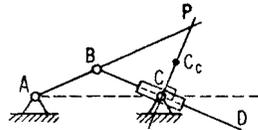


FIG. 3.20 Swinging-block mechanism:  $CC_c = C_cP$ .

### 3.4.4 Bobillier's Theorem

Consider two separate rays, 1 and 2 (Fig. 3.21), with a pair of distinct conjugate points on each,  $A_1, C_1$ , and  $A_2, C_2$ . Let  $Q_{A_1A_2}$

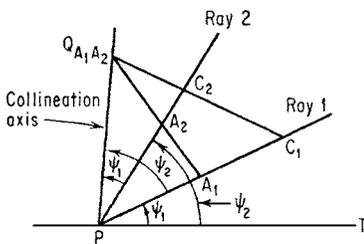


FIG. 3.21 Bobillier's construction.

be the intersection of  $A_1A_2$  and  $C_1C_2$ . Then the line through  $PQ_{A_1A_2}$  is called the "collineation axis," unique for the pair of rays 1 and 2, regardless of the choice of conjugate point pairs on these rays. Bobillier's theorem states that the angle between the common tangent of the centrodes and one ray is equal to the angle between the other ray and the collineation axis, both angles being described in the same sense.<sup>368</sup> Also see Ref. 421f, p. 3.31.

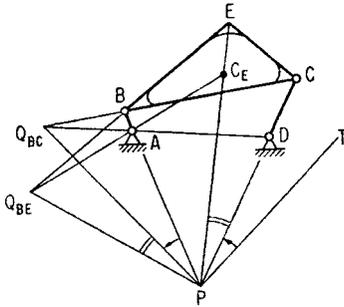
The collineation axis is parallel to the line joining the inflection points on the two rays.

Bobillier's construction for determining the curvature of point-path trajectories is illustrated for two types of mechanisms in

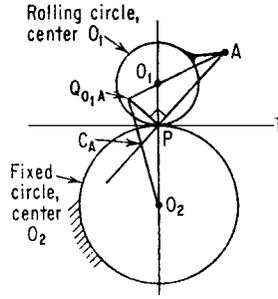
Figs. 3.22 and 3.23.

Another method for finding centers of path curvature is Hartmann's construction, described in Refs. 83 and 421f, pp. 332–336.

Occasionally, especially in the design of linkages with a dwell (temporary rest of output link), one may also use the "sextic of constant curvature," known also as the  $\rho$  curve,<sup>32,421f</sup> the locus of all points in the moving centrode whose paths at a given instant have the same numerical value of the radius of curvature.



**FIG. 3.22** Bobillier's construction for the center of curvature  $C_E$  of path of  $E$  on coupler of four-bar mechanism in position shown.



**FIG. 3.23** Bobillier's construction for cycloidal motion. Determination of  $C_A$ , the center of curvature of the path of  $A$ , attached to the rolling circle (in position shown).

The equation of the  $\rho$  curve in the cartesian coordinate system in which  $PT$  is the positive  $x$  axis and  $PN$  the positive  $y$  axis is

$$(x^2 + y^2)^3 - \rho^2(x^2 + y^2 - \delta y)^2 = 0 \tag{3.20a}$$

where  $\rho$  is the magnitude of the radius of path curvature and  $\delta$  is that of the inflection circle diameter.

**3.4.5 The Cubic of Stationary Curvature (the  $k_u$  Curve)<sup>421f</sup>**

The " $k_u$  curve" is defined as the locus of all points in the moving centrode whose rate of change of path curvature in a given position is zero:  $dp/ds = 0$ . Paths of points on this curve possess "four-point contact" with their osculating circles. Under the same assumptions as in Sec. 3.4.1, the following is the equation of the  $k_u$  curve:

$$(\sin \psi \cos \psi)/r = (\sin \psi)/m + (\cos \psi)l \tag{3.21}$$

where  $(r, \psi) =$  polar coordinates of a point on the  $k_u$  curve

$$m = -3\delta/(d\delta/ds) \tag{3.22}$$

$$l = 3r_p r_\pi / (2r_\pi - r_p)$$

In cartesian coordinates ( $x$  and  $y$  axes  $PT$  and  $PN$ ),

$$(x^2 + y^2)(mx + yl) - lmx y = 0 \tag{3.23}$$

The locus of the centers of curvature of all points on the  $k_u$  curve is known as the "cubic of centers of stationary curvature,"<sup>421f</sup> or the " $k_a$  curve." Its equation is

$$(x^2 + y^2)(mx + l^*y) - l^*mxy = 0 \tag{3.24}$$

where  $l/l^* - 1/l^* = 1/\delta$  (3.25)

The construction and properties of these curves are discussed in Refs. 26, 256, and 421f.

The intersection of the cubic of stationary curvature and the inflection circle yields the "Ball point"  $U(r_u, \psi_u)$ , which describes an approximate straight line, i.e., its path

possesses four-point contact with its tangent (Ref. 421f, pp. 354–356). The coordinates of the Ball point are

$$\psi_u = \tan^{-1} \frac{2r_p - r_\pi}{(r_\pi - r_p)(d\delta/ds)} \tag{3.26}$$

$$r_u = \delta \sin \psi_u \tag{3.27}$$

In the case of a circle rolling inside or outside a fixed circle, the Ball point coincides with the inflection pole.

Technical applications of the cubic of stationary curvature, other than design analysis in general, include the generation of  $n$ -sided polygons,<sup>32</sup> the design of intermittent-motion mechanisms such as the type described in Ref. 426, and approximate straight-line generation. In many of these cases the curves degenerate into circles and straight lines.<sup>32</sup> Special analyses include the “Cardan positions of a plane” (osculating circle of moving centrode inside that of the fixed centrode, one-half its size; stationary inflection-circle diameter)<sup>49,126</sup> and dwell mechanisms. The latter utilize the “ $q_1$  curve” (locus of points having equal radii of path curvature in two distinct positions of the moving centrode) and its conjugate, the “ $q_m$  curve.” See also Ref. 395a.

**3.4.6 Five and Six Infinitesimally Separated Positions of a Plane** (Ref. 421f, pp. 241–245)

In the case of five infinitesimal positions, there are in general four points in the moving plane, called the “Burmester points,” whose paths have “five-point contact” with their osculating circles. These points may be all real or pairwise imaginary. Their application to four-bar motion is outlined in Refs. 32, 411, 469, and 489, and related computer programs are listed in Ref. 129, the last also summarizing the applicable results of six-position theory, insofar as they pertain to four-bar motion. Burmester points and points on the cubic of stationary curvature have been used in a variety of six-link dwell mechanisms.<sup>32,159</sup>

**3.4.7 Application of Curvature Theory to Accelerations** (Ref. 421f, p. 313)

1. The acceleration  $\mathbf{A}_p$  of the instant center (as a point of the moving centrode) is given by  $\mathbf{A}_p = \omega_p^2(\mathbf{PW}_0)$ ; it is the only point of the moving centrode whose acceleration is independent of the angular acceleration  $\alpha_p$ .
2. The inflection circle (also called the “de la Hire circle” in this connection) is the locus of points having zero acceleration normal to their paths.
3. The locus of all points on the moving centrode, whose tangential acceleration (i.e., acceleration along path) is zero, is another circle, the “Bresse circle,” tangent to the principal normal at  $P$ , with diameter equal to  $-\omega_p^2\delta/\alpha_p$  where  $\alpha_p$  is the angular acceleration of the moving centrode, the positive sense of which is the same as that of  $\theta$ . In complex vector form the diameter of the Bresse circle is  $i\omega_p^2\delta/\alpha_p$  (Ref. 421f, pp. 336–338).
4. The intersection of these circles, other than  $P$ , determines the point  $F$ , with zero total acceleration, known as the “acceleration center.” It is located at the intersection of the inflection circle and a ray of angle  $\gamma$ , where

$$\gamma = \angle W_0PF = \tan^{-1}(\alpha_p/\omega_p^2) \quad 0 \leq |\gamma| \leq 90^\circ$$

measured in the direction of the angular acceleration (Ref. 421f, p. 337).

- The acceleration  $\mathbf{A}_B$  of any point  $B$  in the moving system is proportional to its distance from the acceleration center:

$$\mathbf{A}_B = (\mathbf{B}\Gamma)(d^{-\gamma})(\omega_p^4 + \alpha_p^2)^{1/2} \quad (3.28)$$

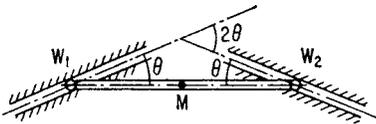
- The acceleration vector  $\mathbf{A}_B$  of any point  $B$  makes an angle  $\gamma$  with the line joining it to the acceleration center [see Eq. (3.28)], where  $\gamma$  is measured from  $\mathbf{A}_B$  in the direction of angular acceleration (Ref. 421*f*, p. 340).
- When the acceleration vectors of two points ( $V, U$ ) on one link, other than the pole, are known, the location of the acceleration center can be determined from item 6 and the equation

$$|\tan \gamma| = \frac{|A_{U/V}^t|}{A_{U/V}^n}$$

- The concept of acceleration centers and images can be extended also to the higher accelerations<sup>41</sup> (see also Sec. 3.3).

### 3.4.8 Examples of Mechanism Design and Analysis Based on Path Curvature

- Mechanism used in guiding the grinding tool in large gear generators (Fig. 3.24): The radius of path curvature  $\rho_m$  of  $M$  at the instant shown:  $\rho_m = (W_1 W_2)/(2 \tan^3 \theta)$ , at which instant  $M$  is on the cubic of stationary curvature belonging to link  $W_1 W_2$ ;  $\rho_m$  is arbitrarily large if  $\theta$  is sufficiently small.

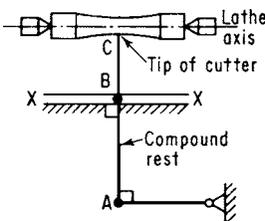


**FIG. 3.24** Mechanism used in guiding the grinding tool in large gear generators. (Due to A. H. Candee, Rochester, N.Y.)  $MW_1 = MW_2$ ; link  $W_1 W_2$  constrained by straight-line guides for  $W_1$  and  $W_2$ .

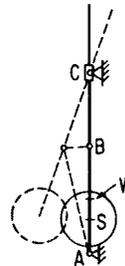
- Machining of radii on tensile test specimens<sup>175,488</sup> (Fig. 3.25):  $C$  lies on cubic of stationary curvature;  $AB$  is the diameter of the inflection circle for the motion of link  $ABC$ ; radius of curvature of path of  $C$  in the position shown:

$$\rho_c = (AC)^2/(BC)$$

- Pendulum with large period of oscillation, yet limited size<sup>283,434</sup> (Fig. 3.26), as used



**FIG. 3.25** Machining of radii on tensile test specimens.  $B$  guided along  $X - X$ .



**FIG. 3.26** Pendulum with large period of oscillation.

in recording ship's vibrations:  $AB = a$ ,  $AC = b$ ,  $CS = s$ ,  $r_t =$  radius of gyration of the heavy mass  $S$  about its center of gravity. If the mass other than  $S$  and friction are negligible, the length  $l$  of the equivalent simple pendulum is given by

$$l = \frac{r_t^2 + s^2}{(b/a)(b - a)} - s$$

where the distance  $CW$  is equal to  $(b/a)(b - a)$ . The location of  $S$  is slightly below the inflection point  $W$ , in order for the oscillation to be stable and slow.

4. Modified geneva drive in high-speed bread wrapper<sup>377</sup> (Fig. 3.27): The driving pin of the geneva motion can be located at or near the Ball point of the pinion motion; the path of the Ball point, approximately square, can be used to give better kinematic characteristics to a four-station geneva than the regular crankpin design, by reducing peak velocities and accelerations.

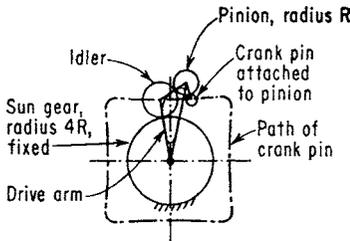


FIG. 3.27 Modified geneva drive in high-speed bread wrapper.

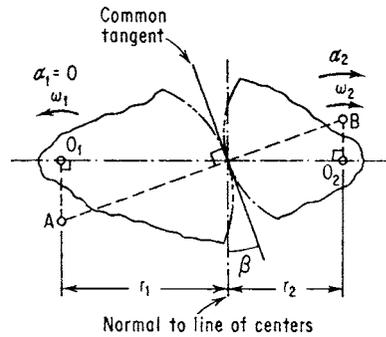


FIG. 3.28 Angular acceleration diagram for noncircular gears.

5. Angular acceleration of noncircular gears (obtainable from equivalent linkage  $O_1ABO_2$ ) (Ref. 116, discussion by A. H. Candee; Fig. 3.28):

Let  $\omega_1 =$  angular velocity of left gear, assumed constant, counterclockwise  
 $\omega_2 =$  angular velocity of right gear, clockwise  
 $\alpha_2 =$  clockwise angular acceleration of right gear

Then 
$$\alpha_2 = [r_1(r_1 + r_2)/r_2^2] (\tan \beta)\omega_1^2$$

**3.5 DIMENSIONAL SYNTHESIS: PATH, FUNCTION, AND MOTION GENERATION**<sup>106,421f</sup>

In the design of automatic machinery, it is often required to guide a part through a sequence of prescribed positions. Such motions can be mechanized by dimensional synthesis based on the kinematic geometry of distinct positions of a plane. In plane motion, a "kinematic plane," hereafter called a "plane," refers to a rigid body, arbitrary in extent. The position of a plane is determined by the location of two of its points,  $A$  and  $B$ , designated as  $A_i, B_i$  in the  $i$ th position.

3.5.1 Two Positions of a Plane

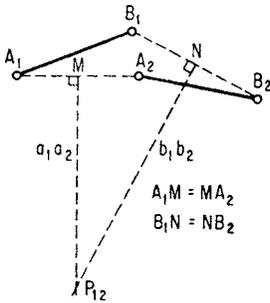


FIG. 3.29 Two positions of a plane. Pole  $P_{12} = a_1a_2 \times b_1b_2$ .

According to “Chasles’s theorem,” the motion from  $A_1B_1$  to  $A_2B_2$  (Fig. 3.29) can be considered *as though* it were a rotation about a point  $P_{12}$ , called the pole, which is the intersection of the perpendicular bisectors  $a_1a_2$ ,  $b_1b_2$  of  $A_1A_2$  and  $B_1B_2$ , respectively.  $A_1, A_2, \dots$ , are called “corresponding positions” of point A;  $B_1, B_2, \dots$ , those of point B;  $A_1B_1, A_2B_2, \dots$ , those of the plane AB.

A similar construction applies to the “relative motion of two planes” (Fig. 3.30)  $AB$  and  $CD$  (positions  $A_iB_i$  and  $C_iD_i$ ,  $i = 1, 2$ ). The “relative pole”  $Q_{12}$  is constructed by transferring the figure  $A_2B_2C_2D_2$  as a rigid body to bring  $A_2$  and  $B_2$  into coincidence with  $A_1$  and  $B_1$ , respectively, and denoting the new positions of  $C_2, D_2$ , by  $C_2^1, D_2^1$ , respectively. Then  $Q_{12}$  is obtained from  $C_1D_1$  and  $C_2^1D_2^1$  as in Fig. 3.29.

Then  $Q_{12}$  is obtained from  $C_1D_1$  and  $C_2^1D_2^1$  as in Fig. 3.29.

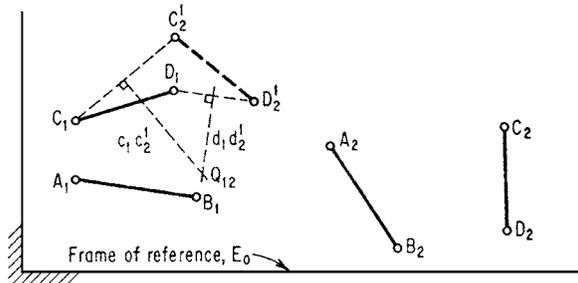


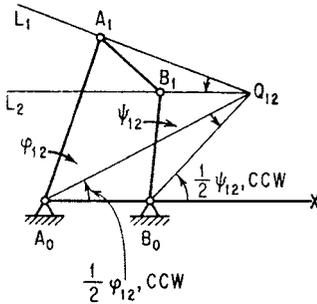
FIG. 3.30 Relative motion of two planes, AB and CD. Relative pole,  $Q_{12} = c_1c_2^1 \times d_1d_2^1$ .

1. The motion of  $A_1B_1$  to  $A_2B_2$  in Fig. 3.29 can be carried out by four-link mechanisms in which A and B are coupler-hinge pivots and the fixed-link pivots  $A_0, B_0$  are located on the perpendicular bisectors  $a_1a_2, b_1b_2$ , respectively.
2. To construct a four-bar mechanism  $A_0ABB_0$  when the corresponding angles of rotation of the two cranks are prescribed (in Fig. 3.31 the construction is illustrated with  $\phi_{12}$  clockwise for  $A_0A$  and  $\psi_{12}$  clockwise for  $B_0B$ ):
  - a. From line  $A_0B_0X$ , lay off angles  $\frac{1}{2}\phi_{12}$  and  $\frac{1}{2}\psi_{12}$  opposite to desired direction of rotation of the cranks, locating  $Q_{12}$  as shown.
  - b. Draw any two straight lines  $L_1$  and  $L_2$  through  $Q_{12}$ , such that

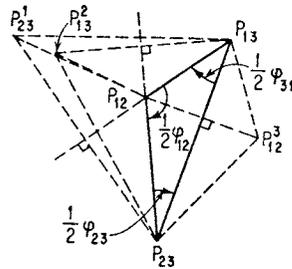
$$\angle L_1Q_{12}L_2 = \angle A_0Q_{12}B_0$$

in magnitude and sense.

- c.  $A_1$  can be located on  $L_1, B_1$ , and  $L_2$ , and when  $A_0A_1$  rotates clockwise by  $\phi_{12}$ ,  $B_0B_1$  will rotate clockwise by  $\psi_{12}$ . Care must be taken, however, to ensure that the mechanism will not lock in an intermediate position.



**FIG. 3.31** Construction of four-bar mechanism  $A_0A_1B_1B_0$  in position 1, for prescribed rotations  $\phi_{12}$  vs.  $\psi_{12}$ , both clockwise in this case.



**FIG. 3.32** Pole triangle for three positions of a plane. Pole triangle  $P_{12}P_{23}P_{31}$  for three positions of a plane; image poles  $P_{12}^3, P_{23}^1, P_{31}^2$ ; subtended angles  $\frac{1}{2}\phi_{23}, \frac{1}{2}\phi_{31}$ .

**3.5.2 Three Positions of a Plane ( $A_iB_i, i = 1, 2, 3$ )<sup>420</sup>**

In this case there are three poles  $P_{12}, P_{23}, P_{31}$  and three associated rotations  $\phi_{12}, \phi_{23}, \phi_{31}$ , where  $\phi_{ij} = \angle A_iP_{ij}A_j = \angle B_iB_jB_j$ . The three poles form the vertices of the *pole triangle* (Fig. 3.32). Note that  $P_{ij} = P_{ji}^*$  and  $\phi_{ij} = -\phi_{ji}$ .

**Theorem of the Pole Triangle.** The internal angles of the pole triangle, corresponding to three distinct positions of a plane, are equal to the corresponding halves of the associated angles of rotation  $\phi_{ij}$  which are connected by the equation

$$\frac{1}{2}\phi_{12} + \frac{1}{2}\phi_{23} + \frac{1}{2}\phi_{31} = 180^\circ \quad \angle \frac{1}{2}\phi_{ij} = \angle P_{ik}P_{ij}P_{jk}$$

Further developments, especially those involving subtention of equal angles, are found in the literature.<sup>32</sup>

For any three corresponding points  $A_1, A_2, A_3$ , the center  $M$  of the circle passing through these points is called a “center point.” If  $P_{ij}$  is considered as though fixed to link  $A_iB_i$  (or  $A_jB_j$ ) and  $A_iB_i$  (or  $A_jB_j$ ) is transferred to position  $k$  ( $A_kB_k$ ), then  $P_{ij}$  moves to a new position  $P_{ij}^k$ , known as the “image pole,” because it is the image of  $P_{ij}$ , reflected about the line joining  $P_{ik}P_{jk}$ .  $\Delta P_{ik}P_{ij}P_{jk}$  is called an “image-pole triangle” (Fig. 3.32). For “circle-point” and “center-point circles” for three finite positions of a moving plane, see Ref. 106, pp. 436–446 and Ref. 421f, pp. 114–122.

**3.5.3 Four Positions of a Plane ( $A_iB_i, i = 1, 2, 3, 4$ )**

With four distinct positions, there are six poles  $P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}$  and four pole triangles ( $(P_{12}P_{23}P_{13}), (P_{12}P_{24}P_{14}), (P_{13}P_{34}P_{14}), (P_{23}P_{34}P_{24})$ ).

Any two poles whose subscripts are all different are called “complementary poles.” For example,  $P_{23}P_{14}$ , or generally  $P_{ij}P_{kl}$  where  $i, j, k, l$  represents any permutation of the numbers 1, 2, 3, 4. Two complementary-pole pairs constitute the two diagonals of a “complementary-pole quadrilateral,” of which there are three:  $(P_{12}P_{24}P_{33}P_{13}), (P_{13}P_{32}P_{24}P_{14})$ , and  $(P_{14}P_{43}P_{32}P_{12})$ .

Also associated with four positions are six further points  $\Pi_{ik}$  found by intersections of opposite sides of complementary-pole quadrilaterals, or their extensions, as follows:  $\Pi_{ik} = P_{il}P_{kl} \times P_{ij}P_{kj}$

**3.5.4 The Center-Point Curve or Pole Curve**<sup>32,67,127,421f</sup>

For three positions, a center point corresponds to *any* set of corresponding points; for four corresponding points to have a common center point, point  $A_1$  can no longer be located arbitrarily in plane  $AB$ . However, a curve exists in the frame of reference called the “center-point curve” or “pole curve,” which is the locus of centers of circles, each of which passes through *four* corresponding points of the plane  $AB$ . The center-point curve may be obtained from any complementary-pole quadrilateral; if associated with positions  $i, j, k, l$ , the center-point curve will be denoted by  $m_{ijkl}$ . Using complex numbers, let  $\mathbf{OP}_{13} = \mathbf{a}$ ,  $\mathbf{OP}_{23} = \mathbf{b}$ ,  $\mathbf{OP}_{14} = \mathbf{c}$ ,  $\mathbf{OP}_{24} = \mathbf{d}$ , and  $\mathbf{OM} = \mathbf{z} = x + iy$ , where  $\mathbf{OM}$  represents the vector from an arbitrary origin  $O$  to a point  $M$  on the center-point curve. The equation of the center-point<sup>127</sup> curve is given by

$$\frac{(\bar{z} - \bar{\mathbf{a}})(\mathbf{z} - \mathbf{b})}{(\mathbf{z} - \mathbf{a})(\bar{z} - \bar{\mathbf{b}})} = \frac{(\bar{z} - \bar{\mathbf{c}})(\mathbf{z} - \mathbf{d})}{(\mathbf{z} - \mathbf{c})(\bar{z} - \bar{\mathbf{d}})} = e^{2i\varphi} \tag{3.29}$$

where  $\varphi = \angle P_{16}MP_{23} = \angle P_{14}MP_{24}$ . In cartesian coordinates with origin at  $P_{12}$ , this curve is given in Ref. 16 by the following equation:

$$(x^2 + y^2)(j_2x - j_1y) + (j_1k_2 - j_2k_1 - j_3)x^2 + (j_1k_2 - j_2k_1 + j_3)y^2 + 2j_4xy + (-j_1k_3 + j_2k_4 + j_3k_1 - j_4k_2)x + (j_1k_4 + j_2k_3 - j_3k_2 - j_4k_1)y = 0 \tag{3.30}$$

where

$$\begin{aligned} k_1 &= x_{13} + x_{24} \\ k_2 &= y_{13} + y_{24} \\ k_3 &= x_{13}y_{24} + y_{13}x_{24} \\ k_4 &= x_{13}x_{24} - y_{13}y_{24} \\ j_1 &= x_{23} + x_{14} - k_1 \\ j_2 &= y_{23} + y_{14} - k_2 \\ j_3 &= x_{23}y_{14} + x_{14}y_{23} - k_3 \\ j_4 &= x_{23}x_{14} - y_{23}y_{14} - k_4 \end{aligned} \tag{3.31}$$

and  $(x_{ij}, y_{ij})$  are the cartesian coordinates of pole  $P_{ij}$ . Equation (3.30) represents a third-degree algebraic curve, passing through the six poles  $P_{ij}$  and the six points  $\prod_{ij}$ . Furthermore, any point  $M$  on the center-point curve subtends equal angles, or angles differing by two right angles, at opposite sides ( $P_{ij}P_{jl}$ ) and ( $P_{ik}P_{kl}$ ) of a complementary-pole quadrilateral, provided the sense of rotation of subtended angles is preserved:

$$\angle P_{ij}MP_{jl} = \angle P_{ik}MP_{kl} \dots \tag{3.32}$$

**Construction of the Center-Point Curve  $m_{ijkl}$** <sup>32</sup> When the four positions of a plane are known ( $A_iB_i, i = 1, 2, 3, 4$ ), the poles  $P_{ij}$  are constructed first; thereafter, the center-point curve is found as follows:

A chord  $P_{ij}P_{jk}$  of a circle, center  $O$ , radius

$$R = \overline{P_{ij}P_{jk}}/2 \sin \theta$$

(Fig. 3.33) subtends the angle  $\theta \pmod{\pi}$  at any point on its circumference. For any value of  $\theta$ ,  $-180^\circ \leq \theta \leq 180^\circ$ , two corresponding circles can be drawn following Fig. 3.33,

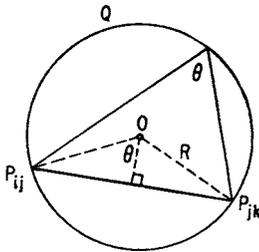


FIG. 3.33 Subtention of equal angles.

using as chords the opposite sides  $P_{ij}P_{jk}$  and  $P_{il}P_{kl}$  of a complementary-pole quadrilateral; intersections of such corresponding circles are points ( $M$ ) on the center-point curve, provided Eq. (3.32) is satisfied.

As a check, it is useful to keep in mind the following angular equalities:

$$\frac{1}{2} \angle A_i M A_l = \angle P_{ij} M P_{jl} = \angle P_{ik} M P_{kl}$$

Also see Ref. 421f, p. 189.

**Use of the Center-Point Curve.** Given four positions of a plane  $A_i B_i$  ( $i = 1, 2, 3, 4$ ) in a coplanar motion-transfer process, we can mechanize the motion by selecting points on the center-point curve as fixed pivots.

**EXAMPLE<sup>91</sup>** A stacker conveyor for corrugated boxes is based on the design shown schematically in Fig. 3.34. The path of  $C$  should be as nearly vertical as possible; if  $A_0, A_1, AC, C_1 C_2 C_3 C_4$  are chosen to suit the specifications,  $B_0$  should be chosen on the center-point curve determined from  $A_i C_i, i = 1, 2, 3, 4$ ;  $B_1$  is then readily determined by inversion, i.e., by drawing the motion of  $B_0$  relative to  $A_1 C_1$  and locating  $B_1$  at the center of the circle thus described by  $B_0$  (also see next paragraphs).

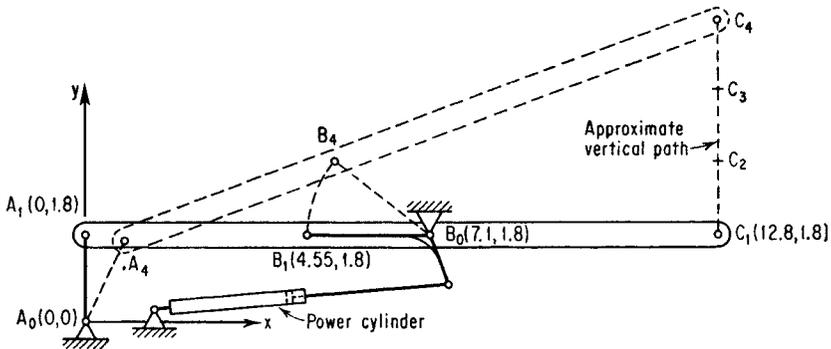


FIG. 3.34 Stacker conveyor drive.

**3.5.5 The Circle-Point Curve**

The circle-point curve is the kinematical inverse of the center-point curve. It is the locus of all points  $K$  in the moving plane whose four corresponding positions lie on one circle. If the circle-point curve is to be determined for positions  $i$  of the plane  $AB$ , Eqs. (3.29), (3.30), and (3.31) would remain unchanged, except that  $P_{jk}, P_{kl}$ , and  $P_{jl}$  would be replaced by the image poles  $P_{jk}^i, P_{kl}^i$ , and  $P_{jl}^i$ , respectively.

The center-point curve lies in the frame or reference plane; the circle-point curve lies in the moving plane. In the above example, point  $B_1$  is on the circle-point curve for plane  $AC$  in position 1. The example can be solved also by selecting  $B_1$  on the circle-point curve in  $A_1 C_1$ ;  $B_0$  is then the center of the circle through  $B_1 B_2 B_3 B_4$ . A computer program for the center-point and circle-point curves (also called “Burmester curves”) is outlined in Refs. 383 and 421f, p. 184.

**SPECIAL CASE** If the corresponding points  $A_1A_2A_3$  lie on a straight line,  $A_1$  must lie on the circle through  $P_{12}P_{13}P_{23}^1$ ; for four corresponding points  $A_1A_2A_3A_4$  on one straight line,  $A_1$  is located at intersection, other than  $P_{12}$ , of circles through  $P_{12}^1P_{13}^1P_{23}^1$  and  $P_{12}^1P_{14}^1P_{24}^1$ , respectively. Applied to straight-line guidance in slider-crank and four-bar drives in Ref. 251; see also Refs. 32 and 421f, pp. 491–494.

**3.5.6 Five Positions of a Plane** ( $A_iB_i, i = 1, 2, 3, 4, 5$ )

In order to obtain accurate motions, it is desirable to specify as many positions as possible; at the same time the design process becomes more involved, and the number of “solutions” becomes more restricted. Frequently four or five positions are the most that can be economically prescribed.

Associated with five positions of a plane are four sets of points  $K_u^{(i)}$  ( $u = 1, 2, 3, 4$  and  $i$  is the position index as before) whose corresponding five positions lie on one circle; to each of these circles, moreover, corresponds a center point  $M_u$ . These circle points  $K_u^{(i)}$  and corresponding center points  $M_u$  are called “Burmester point pairs.” These four point pairs may be all real or pairwise imaginary (all real, two-point pairs real and two point pairs imaginary, or all point pairs imaginary).<sup>127,421f</sup> Note the difference, for historical reasons, between the above definition and that given in Sec. 3.4.6 for infinitesimal motion. The location of the center points,  $M_u$ , can be obtained as the intersections of two center-point curves, such as  $m_{1234}$  and  $m_{1235}$ .

A complex-number derivation of their location,<sup>127,421f</sup> as well as a computer program for simultaneous determination of the coordinates of both  $M_u$  and  $K_u^{(i)}$ , is available.<sup>108,127,380,421f</sup>

An algebraic equation for the coordinates ( $x_u, y_u$ ) of  $M_u$  is given in Ref. 16 as follows. Origin at  $P_{12}$ , coordinates of  $P_{ij}$  are  $x_{ij}, y_{ij}$ :

$$x_u = \frac{(u - \tan \frac{1}{2}\theta_{12})[l_1(k_2 - k_3u) - l_2(e_2 - e_3u)]}{p_1u^2 + p_2u + p_3} \tag{3.33}$$

$$y_u = \frac{(u - \tan \frac{1}{2}\theta_{12})[l_1(k_4 + k_1u) - l_2(e_4 + e_1u)]}{p_1u^2 + p_2u + p_3} \tag{3.34}$$

where  $\tan \frac{1}{2}\theta_{12} = (x_{13}y_{23} - x_{23}y_{13})(x_{13}x_{23} + y_{13}y_{23})$  (3.34)

and  $u$  is a root of  $m_4u^4 + m_3u^3 + m_2u^2 + m_1u + m_0 = 0$

wherein

$$\begin{aligned} m_0 &= p_3(q_1 + l_3p_3) \\ m_1 &= p_2(q_1 + 2l_3p_3)p_2 + q_2p_3 - q_3 \tan \frac{1}{2}\theta_{12} \\ m_2 &= q_0p_3 + q_2p_2 + q_1p_1 + l_3(p_2^2 + 2p_1p_3) - q_5 \tan \frac{1}{2}\theta_{12} + q_3 \\ m_3 &= q_0p_2 + p_1(q_2 + 2l_3p_2) - q_4 \tan \frac{1}{2}\theta_{12} + q_5 \\ m_4 &= p_1(q_0 + l_3p_1) + q_4 \end{aligned} \tag{3.35}$$

$$\begin{aligned} q_0 &= d_1h_3 - d_3h_1 & h_1 &= k_1l_1 - e_1l_2 \\ q_1 &= d_2h_4 - d_4h_2 & h_2 &= k_2l_1 - e_2l_2 \\ q_2 &= -d_1h_2 + d_2h_1 - d_3h_4 + d_4h_3 & h_3 &= k_3l_1 - e_3l_2 \\ q_3 &= h_2^2 + h_4^2 & h_4 &= k_4l_1 - e_4l_2 \end{aligned} \tag{3.36}$$

$$\begin{aligned}
 q_4 &= h_1^2 + h_3^2 & p_1 &= k_3 e_1 - k_1 e_3 \\
 q_5 &= 2(h_1 h_4 - h_2 h_3) & p_2 &= k_3 e_4 + k_1 e_2 - k_2 e_1 - k_4 e_3 \\
 d_1 &= x_{15} + x_{25} & p_3 &= k_4 e_2 - k_2 e_4 \\
 d_2 &= x_{15} - x_{25} & e_1 &= d_1 - x_{13} - x_{23} \\
 d_3 &= y_{15} + y_{25} & e_2 &= d_2 - x_{13} + x_{23} \\
 d_4 &= y_{15} - y_{25} & e_3 &= d_3 - y_{13} - y_{23} \\
 k_1 &= d_1 - x_{14} - x_{24} & e_4 &= d_4 - y_{13} + y_{23} \\
 k_2 &= d_2 - x_{14} + x_{24} & l_1 &= x_{13} x_{23} + y_{13} y_{23} - l_3 \\
 k_3 &= d_3 - y_{14} - y_{24} & l_2 &= x_{14} x_{24} + y_{14} y_{24} - l_3 \\
 k_4 &= d_4 - y_{14} + y_{24} & l_3 &= x_{15} x_{25} + y_{15} y_{25}
 \end{aligned} \tag{3.36}$$

The Burmester point pairs are discussed in Refs. 16, 67, and 127 and extensions of the theory in Refs. 382, 400, and 421f, pp. 211–230. It is suggested that, except in special cases, their determination warrants programmed computation.<sup>108,421f</sup>

*Use of the Burmester Point Pairs.* As in the example of Sec. 3.5.4, the Burmester point pairs frequently serve as convenient pivot points in the design of linked mechanisms. Thus, in the stacker of Sec. 3.5.4, five positions of  $C_i$  could have been specified in order to obtain a more accurately vertical path for  $C$ ; the choice of locations of  $B_0$  and  $B_1$  would then have been limited to at most two Burmester point pairs (since  $A_0A_1$  and  $C_1C_0^\infty$ , prescribed, are also Burmester point pairs).

**3.5.7 Point-Position Reduction**<sup>2,159,194,421f</sup>

“Point-position reduction” refers to a construction for simplifying design procedures involving several positions of a plane. For five positions, graphical methods would involve the construction of two center-point curves or their equivalent. In point-position reduction, a fixed-pivot location, for instance, would be chosen so that one or more poles coincide with it. In the relative motion of the fixed pivot with reference to the moving plane, therefore, one or more of the corresponding positions coincide, thereby reducing the problem to four or fewer positions of the pivot point; the center-point curves, therefore, may not have to be drawn. The reduction in complexity of construction is accompanied, however, by increased restrictions in the choice of mechanism proportions. An exhaustive discussion of this useful tool is found in Ref. 159.

**3.5.8 Complex-Number Methods**<sup>106,123,371,372,380,381,421f,435</sup>

Burmester-point theory has been applied to function generation as well as to path generation and combined path and function generation.<sup>106,127,380,421f</sup> The most general approach to path and function generation in plane motion utilizes complex numbers. The vector closure equations are used for each independent loop of the mechanism for every prescribed position and are differentiated once or several times if velocities, accelerations, and higher rates of change are prescribed. The equations are then solved for the

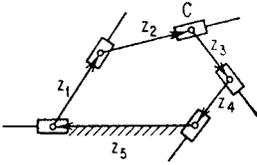


FIG. 3.35 Mechanism derived from a bar-slider chain.

unknown mechanism proportions. This method has been applied to four-bar path and function generators<sup>106,123,127,371,380,384,421f</sup> (the former with prescribed crank rotations), as well as to a variety of other mechanisms. The so-called “path-increment” and “path-increment-ratio” techniques (see below) simplify the mathematics insofar as this is possible. In addition to path and function specification, these methods can take into account prescribed

transmission angles, mechanical advantages, velocity ratios, accelerations, etc., and combinations of these.

Consider, for instance, a chain of links connected by turning-sliding joints (Fig. 3.35). Each bar slider is represented by the vector  $\mathbf{z}_j = r_j e^{i\theta_j}$ . In this case the closure equation for the position shown, and its derivatives are as follows:

Closure: 
$$\sum_{j=1}^5 \mathbf{z}_j = 0$$

Velocity: 
$$\frac{d}{dt} \sum_{j=1}^5 \mathbf{z}_j = 0 \quad \text{or} \quad \sum_{j=1}^5 \lambda_j \mathbf{z}_j = 0$$

where 
$$\lambda_j = (1/r_j)(dr_j/dt) + i(d\theta_j/dt) \quad (t = \text{time})$$

Acceleration: 
$$\frac{d}{dt} \sum_{j=1}^5 \lambda_j \mathbf{z}_j = 0 \quad \text{or} \quad \sum_{j=1}^5 \lambda_j \mu_j \mathbf{z}_j = 0$$

where 
$$\mu_j = \lambda_j + (1/\lambda_j)(d\lambda_j/dt)$$

Similar equations hold for other positions. After suitable constraints are applied on the bar-slider chain (i.e., on  $r_j, \theta_j$ ) in accordance with the properties of the particular type of mechanism under consideration, the equations are solved for the  $\mathbf{z}_j$  vectors, i.e., for the “initial” mechanism configuration.

If the path of a point such as C in Fig. 3.35 (although not necessarily a joint in the actual mechanism represented by the schematic or “general” chain) is specified for a number of positions by means of vectors  $\delta_1, \delta_2, \dots, \delta_k$ , the “path increments” measured from the initial position are  $(\delta_j - \delta_1), j = 2, 3, \dots, k$ . Similarly, the “path increment ratios” are  $(\delta_j - \delta_1)/(\delta_2 - \delta_1), j = 3, 4, \dots, k$ . By working with these quantities, only moving links or their ratios are involved in the computations. The solution of these equations of synthesis usually involves the prior solution of nonlinear “compatibility equations,” obtained from matrix considerations. Additional details are covered in the above-mentioned references. A number of related computer programs for the synthesis of linked mechanisms are described in Refs. 129 and 421f. Numerical methods suitable for such syntheses are described in Ref. 372.

### 3.6 DESIGN REFINEMENT

After the mechanism is selected and its approximate dimensions determined, it may be necessary to refine the design by means of relatively small changes in the proportions, based on more precise design considerations. Equivalent mechanisms and cognates (see Sec. 3.6.6) may also present improvements.

### 3.6.1 Optimization of Proportions for Generating Prescribed Motions with Minimum Error

Whenever mechanisms possess a limited number of independent dimensions, only a finite number of independent conditions can be imposed on their motion. Thus, if a path is to be generated by a point on a linkage (rather than, say, a cam follower), it is not possible—except in special cases—to generate the curve exactly. A desired path (or function) and the actual, or generated, path (or function) may coincide at several points, called “precision points”; between these, the curves differ.

The minimum distance from a point on the ideal path to the actual path is called the “structural error in path generation.” The “structural error in function generation” is defined as the error in the ordinate (dependent variable  $y$ ) for a given value of the abscissa (independent variable  $x$ ). Structural errors exist independent of manufacturing tolerances and elastic deformations and are thus inherent in the design. The combined effect of these errors should not exceed the maximum tolerable error.

The structural error can be minimized by the application of the fundamental theorem of P. L. Chebyshev<sup>16,42</sup> phrased nonrigorously for mechanisms as follows:

If  $n$  independent, adjustable proportions (parameters) are involved in the design of a mechanism, which is to generate a prescribed path or function, then the largest absolute value of the structural error is minimized when there are  $n$  precision points so spaced that the  $n + 1$  maximum values of the structural error between each pair of adjacent precision points—as well as between terminals and the nearest precision points—are numerically equal with successive alterations in sign.

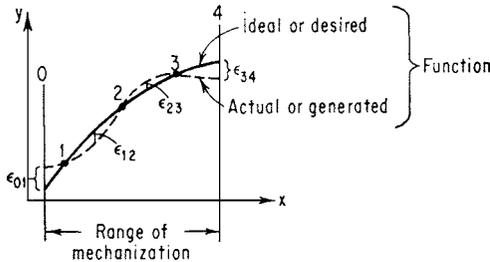


FIG. 3.36 Precision points 1, 2, and 3 and “regions” 01, 12, 23, and 34, in function generation.

In Fig. 3.36 (applied to function generation) the maximum structural error in each “region,” such as 01, 12, 23, and 34, is shown as  $\epsilon_{01}$ ,  $\epsilon_{12}$ ,  $\epsilon_{23}$ , and  $\epsilon_{34}$ , respectively, which represent vertical distances between ideal and generated functions having three precision points. In general, the mechanism proportions and the structural error will vary with the choice of precision points. The spacing of precision points which yields least maximum structural error is called “optimal spacing.” Other definitions and concepts, useful in this connection, are the following:

*n*-point approximation: Generated path (or function) has  $n$  precision points.

*n*th-order approximation: Limiting case of  $n$ -point approximation, as the spacing between precision points approaches zero. In the limit, one precision point is retained, at which point, however, the first  $n - 1$  derivatives, or rates of change of the generated path (or function), have the same values as those of the ideal path (or function).

The following paragraphs apply both to function generation and to planar path generation, provided (in the latter case) that  $x$  is interpreted as the arc length along the

ideal curve and the structural error  $\epsilon_{ij}$  refers to the distance between generated and ideal curves.

**Chebyshev Spacing.**<sup>122</sup> For an  $n$ -point approximation to  $y = f(x)$ , within the range  $x_0 \leq x \leq x_{n+1}$ , Chebyshev spacing of the  $n$  precision points  $x_j$  is given by

$$x_j = \frac{1}{2}(x_0 + x_{n+1}) - \frac{1}{2}(x_{n+1} - x_0) \cos \{ [2j - 1]\pi / 2n \} \quad j = 1, 2, \dots, n$$

Though not generally optimum for finite ranges, Chebyshev spacing often represents a good first approximation to optimal spacing.

The process of respacing the precision points, so as to minimize the maximum structural error, is carried out numerically<sup>122</sup> unless an algebraic solution is feasible.<sup>42,404</sup>

**Respacing of Precision Points to Reduce Structural Error via Successive Approximations.** Let  $x_{ij}^{(1)} = x_j^{(1)} - x_i^{(1)}$ , where  $j = i + 1$ , and let  $x_i^{(1)} (i = 1, 2, \dots, n)$  represent precision-point locations in a first approximation as indicated by the superscript (1). Let  $\epsilon_{ij}^{(1)}$  represent the maximum structural error between points  $x_i^{(1)}, x_j^{(1)}$ , in the first approximation with terminal values  $x_0, x_{n+1}$ . Then a second spacing  $x_{ij}^{(2)} = x_j^{(2)} - x_i^{(2)}$  is sought for which  $\epsilon_{ij}^{(2)}$  values are intended to be closer to optimum (i.e., more nearly equal); it is obtained from

$$x_{ij}^{(2)} = \frac{x_{ij}^{(1)}(x_{n+1} - x_0)}{[\epsilon_{ij}^{(1)}]^m \sum_{i=0}^n \{ x_{ij}^{(1)} / [\epsilon_{ij}^{(1)}]^m \}} \quad (3.37)$$

The value of the exponent  $m$  generally lies between 1 and 3. Errors can be minimized also according to other criteria, for instance, according to least squares.<sup>249</sup> Also see Ref. 404.

**Estimate of Least Possible Maximum Structural Error.** In the case of an  $n$ -point Chebyshev spacing in the range  $x_0 \leq x \leq x_n + 1$  with maximum structural errors  $\epsilon_{ij} (j = i + 1; i = 0, 1, \dots, n)$ ,

$$\epsilon_{\text{opt(estimate)}}^2 = (1/2n)[\epsilon_{01}^2 + \epsilon_{n(n+1)}^2] + (1/n)[\epsilon_{12}^2 + \epsilon_{23}^2 + \dots + \epsilon_{(n-1)n}^2] \quad (3.38)$$

In other spacings different estimates should be used; in the absence of more refined evaluations, the root-mean-square value of the prevailing errors can be used in the general case. These estimates may show whether a refinement of precision-point spacing is worthwhile.

**Chebyshev Polynomials.** Concerning the effects of increasing the number of precision points or changing the range, some degree of information may be gained from an examination of the "Chebyshev polynomials." The Chebyshev polynomial  $T_n(t)$  is that  $n$ th-degree polynomial in  $t$  (with leading coefficient unity) which deviates least from zero within the interval  $\alpha \leq t \leq \beta$ . It can be obtained from the following differential-equation identity by equating to zero coefficients of like powers of  $t$ :

$$2[t^2 - (\alpha + \beta)t + \alpha\beta] T_n''(t) + [2t - (\alpha + \beta)] T_n'(t) - 2n^2 T_n(t) \equiv 0$$

where the primes refer to differentiation with respect to  $t$ . The maximum deviation from zero,  $L_n$ , is given by

$$L_n = (\alpha - \beta)^n / 2^{2n-1}$$

For the interval  $-1 \leq t \leq 1$ , for instance,

$$T_n(t) = (1/2^{n-1}) \cos (n \cos^{-1} t) \quad L_n = 1/2^{n-1}$$

$$T_1(t) = t$$

$$T_2(t) = t^2 - \frac{1}{2}$$

$$T_3(t) = t^3 - (\frac{3}{4})t$$

$$T_4(t) = t^4 - t^2 + \frac{1}{8}$$

.....

Chebyshev polynomials can be used directly in algebraic synthesis, provided the motion and proportions of the mechanism can be suitably expressed in terms of such polynomials.<sup>42</sup>

**Adjusting the Dimensions of a Mechanism for Given Respacing of Precision Points.**

Once the respacing of the precision points is known, it is possible to recompute the mechanism dimensions by a linear computation<sup>122,174,249,446</sup> provided the changes in the dimensions are sufficiently small.

Let  $f(x)$  = ideal or desired functional relationship.

$g(x) = g(x, p_0^{(1)}, p_1^{(1)}, \dots, p_{n-1}^{(1)})$   
 = generated functional relationship in terms of mechanism parameters or proportions  $p_j^{(k)}$ , where  $p_j^{(k)}$  refers to the  $j$ th parameter in the  $k$ th approximation.

$\epsilon^{(1)}(x_i^{(2)})$  = value of structural error at  $x_i^{(2)}$  in the first approximation, where  $x_i^{(2)}$  is a new or respaced location of a precision point, such that ideally  $\epsilon^{(2)}(x_i^{(2)}) = 0$  (where  $\epsilon = f - g$ ).

Then the new values of the parameters  $p_j^{(2)}$  can be computed from the equations

$$\epsilon^{(1)}(x_i^{(2)}) = \sum_{j=0}^{n-1} \frac{\partial g(x_i^{(2)})}{\partial p_j^{(1)}} (p_j^{(2)} - p_j^{(1)}) \quad i = 1, 2, \dots, n \quad (3.39)$$

These are  $n$  linear equations, one each at the  $n$  “precision” points  $x_i^{(2)}$  in the  $n$  unknowns  $p_j^{(2)}$ . The convergence of this procedure depends on the appropriateness of neglecting higher-order terms in Eq. (3.39); this, in turn, depends on the functional relationship and the mechanism and cannot in general be predicted. For related investigations, see Refs. 131 and 184; for respacing via automatic computation and for accuracy obtainable in four-bar function generators, see Ref. 122, and in geared five-bar function generators, see Ref. 397.

**3.6.2 Tolerances and Precision**<sup>17,147,158,174,228,243,482</sup>

After the structural error is minimized, the effects of manufacturing errors still remain.

The accuracy of a motion is frequently expressed as a percentage defined as the maximum output error divided by total output travel (range).

For a general discussion of the various types of errors, see Ref. 482.

Machining errors may cause changes in link dimensions, as well as clearances and backlash. Correct tolerancing requires the investigation of both. If the errors in link dimensions are small compared with the link lengths, their effect on displacements, velocities, and accelerations can be determined by a linear computation, using only first-order terms.

The effects of clearances in the joints and of backlash are more complicated and, in addition to kinematic effects, are likely to affect adversely the dynamic behavior of the mechanism.<sup>147</sup> The kinematic effect manifests itself as an uncertainty in displacements, velocities, accelerations, etc., which, in the absence of load reversal, can be computed as though due to a change in link length, equivalent to the clearance or backlash involved. The dynamic effects of clearances in machinery have been investigated in Ref. 98 to 100.

Since the effect of tolerances will depend on the mechanism and on the "location" of the tolerance in the mechanism, each tolerance should be specified in accordance with the magnitude of its effect on the pertinent kinematic behavior.

### 3.6.3 Harmonic Analysis (see also Sec. 3.9 and bibliography in Ref. 493)

It is sometimes desirable to express the motion of a machine part as a Fourier series in terms of driving motion, in order to analyze dynamic characteristics and to ensure satisfactory performance at high speeds. Harmonic analysis, for example, is used in computing the inertia forces in slider-crank mechanisms in internal-combustion engines<sup>39,367</sup> and also in other mechanisms.<sup>128,286,289,493</sup>

Generally, two types of investigations arise:

1. Determination of the "harmonics" in the motion of a given mechanism as a check on inertial loads and critical speeds
2. Proportioning to minimize higher harmonics<sup>128</sup>

### 3.6.4 Transmission Angles (see also Sec. 3.2.10)<sup>134-136,155,467</sup>

In mechanisms with varying transmission angles  $\mu$ , the optimum design involves the minimization of the deviation  $\nu$  of the transmission angle from its ideal value. Such a design maximizes the force tending to turn the driven link while minimizing frictional resistance, assuming quasi-static operation.

In plane crank-and-rocker linkages, the minimization of the maximum deviation of the transmission angle from  $90^\circ$  has been worked out for given rocker swing angle  $\psi$  and corresponding crank rotation  $\phi$ . In the special case of centric crank-and-rocker linkages ( $\phi = 180^\circ$ ) the solution is relatively simple:  $a^2 + b^2 = c^2 + d^2$  (where  $a$ ,  $b$ ,  $c$ , and  $d$  denote the lengths of crank, coupler, rocker, and fixed link, respectively). This yields  $\sin \nu = (ab/cd)$ ,  $\mu_{\max} = 90^\circ + \nu$ ,  $\mu_{\min} = 90^\circ - \nu$ . The solution for the general case ( $\phi$  arbitrary), including additional size constraints, can be found in Refs. 134 to 136, 155, and 371, and depends on the solution of a cubic equation.

### 3.6.5 Design Charts

To save labor in the design process, charts and atlases are useful when available. Among these are Refs. 199 and 210 in four-link motion; the VDI-Richtlinien Duesseldorf (obtainable through Beuth-Vertrieb GmbH, Berlin), such as 2131, 2132 on the offset turning block and the offset slider crank, and 2125, 2126, 2130, 2136 on the offset slider-crank and crank-and-rocker mechanisms; 2123, 2124 on four-bar mechanisms; 2137 on the in-line swinging block; and data sheets in the technical press.

3.6.6 Equivalent and “Substitute” Mechanisms<sup>106,112,421g</sup>

Kinematic equivalence is explained in Sec. 3.2.12. Ways of obtaining equivalent mechanisms include (1) pin enlargement, (2) kinematic inversion, (3) use of centroids, (4) use of curvature constructions, (5) use of pantograph devices, (6) use of multigeneration properties, (7) substitution of tapes, racks, and chains for rigid links<sup>84,103,159,189,285</sup> and other ways depending on the inventiveness of the designer.\* Of these, (5) and (6) require additional explanation.

The “pantograph” can be used to reproduce a given motion, unchanged, enlarged, reduced, or rotated. It is based on “Sylvester’s plagiograph,” shown in Fig. 3.37.

$AODC$  is a parallelogram linkage with point  $O$  fixed with two similar triangles  $ACC_1$ ,  $DBC$ , attached as shown. Points  $B$  and  $C_1$  will trace similar curves, altered in the ratio  $OC_1/OB = AC_1/AC$  and rotated relative to each other by an amount equal to the angle  $\alpha$ . The ordinary pantograph is the special case obtained when  $B, D, C$ , and  $C, A, C_1$  are collinear. It is used in engraving machines and other motion-copying devices.

Roberts’ theorem<sup>32,182,288,347,421f</sup> states that there are three different but related four-bar mechanisms generating the same coupler curve (Fig. 3.38): the “original”  $ABCDE$ , the “right cognate”  $LKGDE$ , and the “left cognate”  $LHFAE$ . Similarly, slider-crank mechanisms have one cognate each.<sup>182</sup>

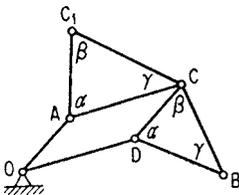


FIG. 3.37 Sylvester’s plagiograph or skew pantograph.

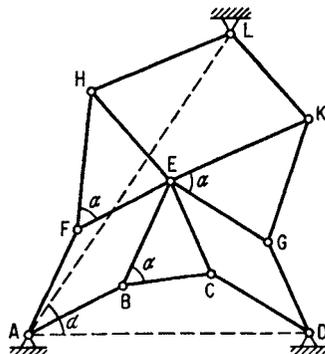
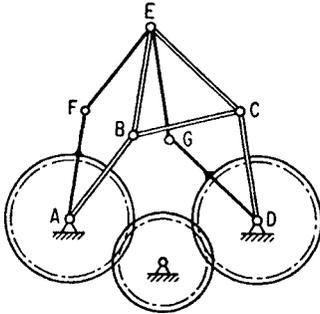


FIG. 3.38 Roberts’ theorem.  $\triangle BEC \approx \triangle FHE \approx \triangle EKG \approx \triangle ALD \approx \triangle AHC \approx \triangle BKD \approx \triangle FLG$ ;  $AFEB$ ,  $EGDC$ ,  $HLKE$  are parallelograms.

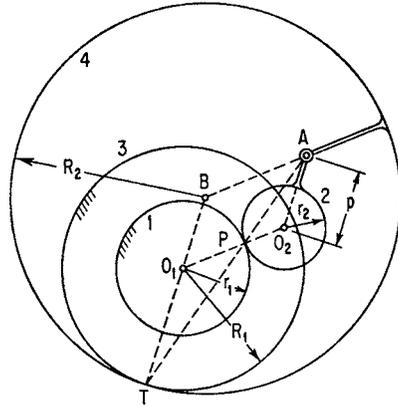
If the “original” linkage has poor proportions, a cognate may be preferable. When Grashof’s inequality is obeyed (Sec. 3.9) and the original is a double rocker, the cognates are crank-and-rocker mechanisms; if the original is a drag link, so are the cognates; if the original does not obey Grashof’s inequality, neither do the cognates, and all three are either double rockers or folding linkages. Several well-known straight-line guidance devices (Watt and Evans mechanisms) are cognates.

*G geared five-bar mechanisms* (Refs. 95, 119, 120, 347, 372, 391, 397, 421f) may also be used to generate the coupler curve of a four-bar mechanism, possibly with better transmission angles and proportions, as, for instance, in the drive of a deep-draw press. The gear ratio in this case is 1:1 (Fig. 3.39), where  $ABCDE$  is the four-bar linkage and  $AFEGD$  is the five-bar mechanism with links  $AF$  and  $GD$  geared to each other by 1:1 gearing. The path of  $E$  is identical in both mechanisms.

\*Investigation of enumeration of mechanisms based on degree-of-freedom requirements are found in Refs. 106, 159, and 162 to 165 with application to clamping devices, tools, jigs, fixtures, and vise jaws.



**FIG. 3.39** Four-bar linkage  $ABCDE$  and equivalent 1:1 geared five-bar mechanism  $AFEGD$ ;  $AFEB$  and  $DGEC$  are parallelograms.



**FIG. 3.40** Double generation of a cycloidal path. For the case shown  $O_1O_2$  and  $AO_2$  rotate in the same direction.  $R_2/R_1 = r_2/r_1 + 1$ ;  $R_1 = p(r_1/r_2)$ ;  $R_2 = p[1 + (r_1/r_2)]$ . Radius ratios are considered positive or negative depending on whether gearing is internal or external.

In Fig. 3.38 each cognate has one such derived geared five-bar mechanism (as in Fig. 3.39), thus giving a choice of six different mechanisms for the generation of any one coupler curve.

**Double Generation of Cycloidal Curves.**<sup>315,385,386</sup> A given cycloidal motion can be obtained by two different pairs of rolling circles (Fig. 3.40). Circle 2 rolls on fixed circle 1 and point  $A$ , attached to circle 2, describes a cycloidal curve. If  $O_1, O_2$  are centers of circles 1 and 2,  $P$  their point of contact, and  $O_1O_2AB$  a parallelogram, circle 3, which is also fixed, has center  $O_1$  and radius  $O_1T$ , where  $T$  is the intersection of extensions of  $O_1B$  and  $AP$ ; circle 4 has center  $B$ , radius  $BT$ , and rolls on circle 3. If point  $A$  is now rigidly attached to circle 4, its path will be the same as before. Dimensional relationships are given in the caption of Fig. 3.40. For analysis of cycloidal motions, see Refs. 385, 386, and 492.

Equivalent mechanisms obtained by multigeneration theory may yield patentable devices by producing “unexpected” results, which constitutes one criterion of patentability. In one application, cycloidal path generation has been used in a speed reducer.<sup>48,426,495</sup> Another form of “cycloidal equivalence” involves adding an idler gear to convert from, say, internal to external gearing; applied to resolver mechanism in Ref. 357.

**3.6.7 Computer-Aided Mechanisms Design and Optimization** (Refs. 64–66, 78, 79, 82, 106, 108, 109, 185, 186, 200, 217, 218, 246, 247, 317, 322–324, 366, 371, 380, 383, 412–414, 421a, 430, 431, 445, 457, 465, 484, 485)

General mechanisms texts with emphasis on computer-aided design include Refs. 106, 186, 323, 421f, 431, 445. Computer codes having both kinematic analysis and synthesis capability in linkage design include KINSYN<sup>217,218</sup> and LINCAGES.<sup>108</sup> Both codes also include interactive computer graphics features. Codes which can perform both kinematic and dynamic analysis for a large class of mechanisms include DRAM and

ADAMS,<sup>64-66,457</sup> DYMARC,<sup>322,324</sup> IMP,<sup>430,465</sup> kinetoelastodynamic codes,<sup>109,421f</sup> codes for the sensitivity analysis and optimization of mechanisms with intermittent-motion elements,<sup>185,186,200</sup> heuristic codes,<sup>78,79,246,247</sup> and many others.<sup>82,317,322,366,484,485</sup>

The variety of computational techniques is as large as the variety of mechanisms. For specific mechanisms, such as cams and gears, specialized codes are available.

In general, computer codes are capable of analyzing both simple and complex mechanisms. As far as synthesis is concerned the situation is complicated by the non-linearity of the motion parameters in many mechanisms and by the impossibility of limiting most motions to small displacements. For the simpler mechanisms synthesis codes are available. For more complex mechanisms parameter variation of analysis codes or heuristic methods are probably the most powerful currently available tools. The subject remains under intensive development, especially with regard to interactive computer graphics [for example, CADSPAM, *computer-aided design of spatial mechanisms* (Ref. 421a)].

### 3.6.8 Balancing of Linkages

At high speeds the inertia forces associated with the moving links cause shaking forces and moments to be transmitted to the frame. Balancing can reduce or eliminate these. An introduction is found in Ref. 421f. (See also Refs. to BAL 25-30, 191, 214, 219, 258-260, 379a, 452, 452a, 452b, 459, 460.)

### 3.6.9 Kinetoelastodynamics of Linkage Mechanisms

Load and inertia forces may cause cyclic link deformations at high speeds, which change the motion of the mechanism and cannot be neglected. An introduction and copious list of references are found in Ref. 421f. (See also Refs. 53, 107, 202, 416, 417.)

## 3.7 THREE-DIMENSIONAL MECHANISMS<sup>5,21,35,421f</sup>

(Sec. 3.9)

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Three-dimensional mechanisms are also called "spatial mechanisms." Points on these mechanisms move on three-dimensional curves. The basic three-dimensional mechanisms are the "spherical four-bar mechanisms" (Fig. 3.41) and the "offset" or "spatial four-bar mechanism" (Fig. 3.42).

The spherical four-bar mechanism of Fig. 3.41 consists of links  $AB$ ,  $BC$ ,  $CD$ , and  $DA$ , each on a great circle of the sphere with center  $O$ ; turning joints at  $A$ ,  $B$ ,  $C$ , and  $D$ , whose axes intersect at  $O$ ; lengths of links measured by great-circle arcs or angles  $\alpha_i$  subtended at  $O$ . Input  $\theta_2$ , output  $\theta_1$ ; single degree of freedom, although  $\sum f_i = 4$  (see Sec. 3.2.2).

Figure 3.42 shows a spatial four-bar mechanism; turning joint at  $D$ , turn-slide (also called cylindrical) joints at  $B$ ,  $C$ , and  $D$ ;  $a_{ij}$  denote minimum distances between axes of joints; input  $\theta_2$  at  $D$ ; output at  $A$  consists of translation  $s$  and rotation  $\theta_1$ ;  $\sum f_i = 7$ ; freedom,  $F = 1$ .

Three-dimensional mechanisms used in practice are usually special cases of the above two mechanisms. Among these are Hooke's joint (a spherical four-bar, with  $\alpha_2 = \alpha_3 = \alpha_4 = 90^\circ$ ,  $90^\circ < \alpha_1 \leq 180^\circ$ ), the wobble plates ( $\alpha_4 < 90^\circ$ ,  $\alpha_2 = \alpha_3 = \alpha_1 = 90^\circ$ ), the space crank,<sup>332</sup> the spherical slider crank,<sup>304</sup> and other mechanisms, whose analysis is outlined in Sec. 3.9.

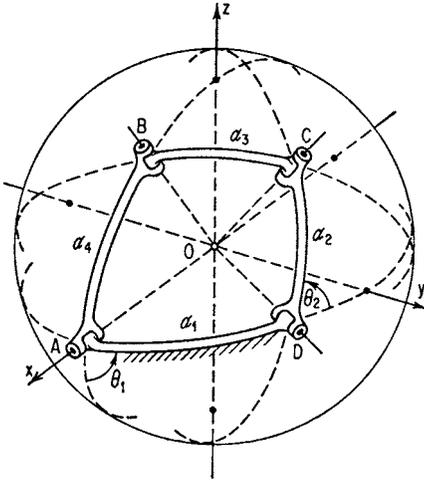


FIG. 3.41 Spherical four-bar mechanism.

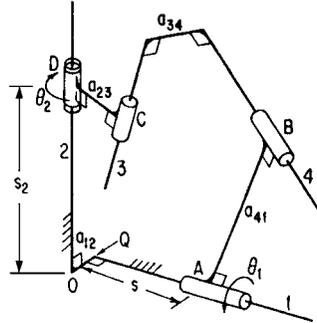


FIG. 3.42 Offset or spatial four-bar mechanism.

The analysis and synthesis of spatial mechanisms require special mathematical tools to reduce their complexity. The analysis of displacements, velocities, and accelerations of the general spatial chain (Fig. 3.42) is conveniently accomplished with the aid of dual vectors,<sup>421c</sup> numbers, matrices, quaternions, tensors, and Cayley-Klein parameters.<sup>85,87,494</sup> The spherical four-bar (Fig. 3.41) can be analyzed the same way, or by spherical trigonometry.<sup>34</sup> A computer program by J. Denavit and R. S. Hartenberg<sup>129</sup> is available for the analysis and synthesis of a spatial four-link mechanism whose terminal axes are nonparallel and nonintersecting, and whose two moving pivots are ball joints. See also Ref. 474 for additional spatial computer programs. For the simpler problems, for verification of computations and for visualization, graphical layouts are useful.<sup>21,33,36,38,462</sup>

Applications of three-dimensional mechanisms involve these motions:

1. Combined translation and rotation (e.g., door openers to lift and slide simultaneously<sup>3</sup>)
2. Compound motions, such as in paint shakers, mixers, dough-kneading machines and filing<sup>8,35,36,304</sup>
3. Motions in shaft couplings, such as universal and constant-velocity joints<sup>4,6,21,35,262</sup> (see Sec. 3.9)
4. Motions around corners and in limited space, such as in aircraft, certain wobble-plate engines, and lawn mowers<sup>60,310,332</sup>
5. Complex motions, such as in aircraft landing gear, remote-control handling devices,<sup>71,270</sup> and pick-and-place devices in automatic assembly machines

When the motion is constrained ( $F = 1$ ), but  $\sum f_i < 7$  (such as in the mechanism shown in Fig. 3.41), any elastic deformation will tend to cause binding. This is not the case when  $\sum f_i = 7$ , as in Fig. 3.42, for instance. Under light-load, low-speed conditions, however, the former may represent no handicap.<sup>10</sup> The “degenerate” cases, usually associated with parallel or intersecting axes, are discussed more fully in Refs. 3, 10, 143, and 490.

In the analysis of displacements and velocities, extensions of the ideas used in plane kinematic analysis have led to the notions of the “instantaneous screw axis,”<sup>738</sup> valid for displacements and velocities; to spatial Euler-Savary equations; and to concepts involving line geometry.<sup>224</sup>

Care must be taken in designing spatial mechanisms to avoid binding and low mechanical advantages.

### 3.8 CLASSIFICATION AND SELECTION OF MECHANISMS

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In this section, mechanisms and their components are grouped into three categories:

- A. Basic mechanism components, such as those adapted for latching, fastening, etc.
- B. Basic mechanisms: the building blocks in most mechanism complexes.
- C. Groups or assemblies of mechanisms, characterized by one or more displacement-time schedules, sequencing, interlocks, etc.; these consist of combinations from categories A and B and constitute important mechanism units or independent portions of entire machines.

Among the major collections of mechanisms and mechanical movements are the following:

- 1M. Barber, T. W.: “The Engineer’s Sketch-Book,” Chemical Publishing Company, Inc., New York, 1940.
- 2M. Beggs, J. S.: “Mechanism,” McGraw-Hill Book Company, Inc., New York, 1955.
- 3M. Hain, K.: “Die Feinwerktechnik,” Fachbuch-Verlag, Dr. Pfanneberg & Co., Giessen, Germany, 1953.
- 4M. “Ingenious Mechanisms for Designers and Inventors,” vols. 1–2, F. D. Jones, ed.: vol. 3, H. L. Horton, ed.; The Industrial Press, New York, 1930–1951.
- 5M. Rauh, K.: “Praktische Getriebelehre,” Springer-Verlag OHG, Berlin, vol. I, 1951; vol. II, 1954.

There are, in addition, numerous others, as well as more special compilations, the vast amount of information in the technical press, the AWF publications,<sup>468</sup> and (as a useful reference in depth), the Engineering Index. For some mechanisms, especially the more elementary types involving fewer than six links, a systematic enumeration of kinematic chains based on degrees of freedom may be worthwhile,<sup>162–165,421c</sup> particularly if questions of patentability are involved. Mechanisms are derived from the kinematic chains by holding one link fixed and possibly by using equivalent and substitute mechanisms (Sec. 3.6.6). The present state of the art is summarized in Ref. 159.

In the following list of mechanisms and components, each item is classified according to category (A, B, or C) and is accompanied by references, denoting one or more of the above five sources, or those at the end of this chapter. In using this listing, it is to be remembered that a mechanism used in one application may frequently be employed in a completely different one, and sometimes combinations of several mechanisms may be useful.

The categories A, B, C, or their combinations are approximate in some cases, since it is often difficult to determine a precise classification.

Adjustments, fine (A, 1M)(A, 2M)(A, 3M)

Adjustments, to a moving mechanism (A, 2M)(AB, 1M); see also Transfer, power

- Airplane instruments and linkages (C, 3M)<sup>342</sup>
- Analog computing mechanisms;<sup>306</sup> see also Computing mechanisms
- Anchoring devices (A, 1M)
- Automatic machinery, special-purpose;<sup>161</sup> automatic handling<sup>363</sup>
- Ball bearings, guides and slides (A, 3M)
- Ball-and-socket joints (A, 1M); see also Joints
- Band drives (B, 3M); see also Tapes
- Bearings (A, 3M)(A, 1M); jewel;<sup>245</sup> for oscillating motion<sup>56</sup>
- Belt gearing (B, 1M)
- Bolts (A, 1M)
- Brakes (B, 1M)
- Business machines, bookkeeping and records (C, 3M)
- Calculating devices (C, 3M);<sup>277</sup> see also Mathematical instruments
- Cameras (C, 3M); see also Photographic devices
- Cam-link mechanisms (BC, 1M)(BC, 5M)<sup>421</sup>
- Cams and cam drives (BC, 4M)(BC, 5M)(BC, 1M)<sup>375</sup>
- Carriages and cars (BC, 1M)
- Centrifugal devices (BC, 1M)
- Chain drives (B, 1M)(B, 5M)<sup>265</sup>
- Chucks, clamps, grips, holders (A, 1M)
- Circular-motion devices (B, 1M)
- Clock mechanisms;<sup>18</sup> see also Escapements (Ref. 106 and 421f, pp. 37–39)
- Clutches, overrunning (BC, 1M)(C, 4M); see also Couplings and clutches
- Computing mechanisms (BC, 5M)<sup>80,271,338,446</sup>
- Couplings and clutches (B, 1M)(B, 3M)(B, 5M);<sup>139,140,225,261,336</sup> see also Joints
- Covers and doors (A, 1M)
- Cranes (AC, 1M)<sup>77</sup>
- Crank and eccentric gear devices (BC, 1M)
- Crushing and grinding devices (BC, 1M)
- Curve-drawing devices (BC, 1M); see also Writing instruments and Mathematical instruments
- Cushioning devices (AC, 1M)
- Cutting devices (A, 1M)<sup>329,354</sup>
- Derailleurs or derailers (see Speed-changing mechanisms) (Refs. 106 and 421f, pp. 27)
- Detents (A, 3M)
- Differential motions (C, 1M)(C, 4M)<sup>188</sup>
- Differentials (B, 5M)<sup>4</sup>
- Dovetail slides (A, 3M)
- Drilling and boring devices (AC, 1M)
- Driving mechanisms for reciprocating parts (C, 3M)
- Duplicating and copying devices (C, 3M)

Dwell linkages (C, 4M)  
 Ejecting mechanisms for power presses (C, 4M)  
 Elliptic motions (B, 1M)  
 Energy storage, instruments and mechanisms involving<sup>434</sup>  
 Energy transfer mechanism, special-purpose<sup>267</sup>  
 Engines, rotary (BC, 1M)  
 Engines, types of (C, 1M)  
 Escapements (B, 2M); see also Ratchets and Clock mechanisms  
 Expansion and contraction devices (AC, 1M)  
 Fasteners<sup>244,423</sup>  
 Feed gears (BC, 1M)  
 Feeding, magazine and attachments (C, 4M)<sup>170</sup>  
 Feeding mechanisms, automatic (C, 4M)(C, 5M)  
 Filtering devices (AB, 1M)<sup>230</sup>  
 Flexure pivots<sup>103,141,269</sup>  
 Flight-control linkages<sup>365</sup>  
 Four-bar chains, mechanisms and devices (B, 5M)<sup>106,112,421f</sup>  
 Frames, machine (A, 1M)  
 Friction gearing (BC, 1M)  
 Fuses (see Escapements)  
 Gears (B, 3M)(B, 1M)  
 Gear mechanisms (BC, 1M)<sup>106</sup>  
 Geneva; see Intermittent motions  
 Geodetic instruments (C, 3M)  
 Governing and speed-regulating devices (BC, 1M)<sup>461</sup>  
 Guidance, devices for (BC, 5M)  
 Guides (A, 1M)(A, 3M)  
 Handles (A, 1M)  
 Harmonic drives<sup>101</sup>  
 High-speed design;<sup>47</sup> special application<sup>339</sup>  
 Hinges (A, 1M); see also Joints  
 Hooks (A, 1M)  
 Hoppers, for automatic machinery (C, 4M), and hopper-feeding devices<sup>234,235,236,421f</sup>  
 Hydraulic converters (BC, 1M)  
 Hydraulic and link devices<sup>89,90,92,168,337</sup>  
 Hydraulic transmissions (C, 4M)  
 Impact devices (BC, 1M)  
 Indexing mechanisms (B, 2M); see also Sec. 3.9 and Intermittent motions  
 Indicating devices (AC, 1M); speed (C, 1M)  
 Injectors, jets, nozzles (A, 1M)  
 Integrators, mechanical<sup>358</sup>

Interlocks (C, 4M)  
Intermittent motions, general;<sup>44,357,468</sup> see also Sec. 3.9  
Intermittent motions from gears and cams (C, 4M)<sup>269</sup>  
Intermittent motions, geneva types (BC, 4M)(BC, 5M);<sup>211,357</sup> see also Indexing mechanisms  
Intermittent motions from ratchet gearing (C, 1M)(C, 4M)  
Joints, all types (A, 1M);<sup>239</sup> see also Couplings and clutches and Hinges  
Joints, ball-and-socket<sup>69</sup>  
Joints, to couple two sliding members (B, 2M)  
Joints, intersecting shafts (B, 2M)  
Joints, parallel shafts (B, 2M)  
Joints, screwed or bolted (A, 3M)  
Joints, skew shafts (B, 2M)  
Joints, soldered, welded, riveted (A, 3M)  
Joints, special-purpose, three-dimensional<sup>272</sup>  
Keys (A, 1M)  
Knife edges (A, 3M)<sup>141</sup>  
Landing gear, aircraft<sup>71</sup>  
Levers (A, 1M)  
Limit switches<sup>498</sup>  
Link mechanisms (BC, 5M)  
Links and connecting rods (A, 1M)  
Locking devices (A, 1M)(A, 3M)(A, 5M)  
Lubrication devices (A, 1M)  
Machine shop, measuring devices (C, 3M)  
Mathematical instruments (C, 3M); see also Curve-drawing devices and Calculating devices  
Measuring devices (AC, 1M)  
Mechanical advantage, mechanisms with high value of (BC, 1M)  
Mechanisms, accurate;<sup>482</sup> general<sup>21,96,153,159,176,180,193,263,278</sup>  
Medical instruments (C, 3M)  
Meteorological instruments (C, 3M)  
Miscellaneous mechanical movements (BC, 5M)(C, 4M)  
Mixing devices (A, 1M)  
Models, kinematic, construction of<sup>51,183</sup>  
Noncircular gearing (Sec. 3.9)  
Optical instruments (C, 3M)  
Oscillating motions (B, 2M)  
Overload-relief mechanisms (C, 4M)  
Packaging techniques, special-purpose<sup>197</sup>  
Packings (A, 1M)

Photographic devices (C, 3M);<sup>478,479,486</sup> see also Cameras  
 Piping (A, 1M)  
 Pivots (A, 1M)  
 Pneumatic devices<sup>57</sup>  
 Press fits (A, 3M)  
 Pressure-applying devices (AB, 1M)  
 Prosthetic devices<sup>311,344,345</sup>  
 Pulleys (AB, 1M)  
 Pumping devices (BC, 1M)  
 Pyrotechnic devices (C, 3M)  
 Quick-return motions (BC, 1M)(C, 4M)  
 Raising and lowering, including hydraulics (BC, 1M)<sup>88,340</sup>  
 Ratchets, detents, latches (AB, 2M)(B, 5M);<sup>18,362,468</sup> see also Escapements  
 Ratchet motions (BC, 1M)<sup>468</sup>  
 Reciprocating mechanisms (BC, 1M)(B, 2M)(BC, 4M)  
 Recording mechanisms, illustrations of;<sup>55,203</sup> recording systems<sup>206</sup>  
 Reducers, speed; cycloidal;<sup>48,331,495</sup> general<sup>308</sup>  
 Releasing devices and circuit breakers<sup>84,325,458</sup>  
 Remote-handling robots;<sup>270</sup> qualitative description<sup>364</sup>  
 Reversing mechanisms, general (BC, 1M)(C, 4M)  
 Reversing mechanisms for rotating parts (BC, 1M)(C, 4M)  
 Robots and manipulators (See Sec. 5.9.10)<sup>421f</sup>  
 Rope drives (BC, 1M)  
 Safety devices, automatic (A, 1M)(C, 4M)<sup>20,481</sup>  
 Screening and sifting (A, 1M)  
 Screw mechanisms (BC, 1M) (See Ref. 468, no. 6071)  
 Screws (B, 5M)  
 Seals, hermetic;<sup>59</sup> O-ring;<sup>209</sup> with gaskets;<sup>46,113,346</sup> multistage<sup>422</sup>  
 Self-adjusting links and slides (C, 4M)  
 Separating and concentrating devices (BC, 1M)  
 Sewing machines (C, 3M)  
 Shafts (A, 1M)(A, 3M); flexible<sup>198,241</sup>  
 Ship instruments (C, 3M)  
 Slider-crank mechanisms (B, 5M)  
 Slides (A, 1M)(A, 3M)  
 Snap actions (A, 2M)  
 Sound, devices using (B, 1M)  
 Spacecraft, mechanical design of<sup>497</sup>  
 Spanners (A, 1M)  
 Spatial body guidance (Refs. 421c, 421d, 421e, 421f)  
 Spatial function generators with higher pairs (Ref. 421b)

Speed-changing mechanisms (C, 4M); also see Transmissions  
 Spindles and centers (A, 1M)  
 Springs (A, 1M)(A, 3M); devices;<sup>81,434</sup> fastening of<sup>343</sup>  
 Springs and mechanisms (BC, 5M)  
 Steering mechanisms (BC, 5M)<sup>254,447,495</sup>  
 Stop mechanisms (C, 4M)  
 Stops (A, 2M)<sup>343</sup>  
 Straight-line motions, guides, parallel motions and devices (B, 1M)(BC, 4M),<sup>72,150,157,476</sup> Sec. 3.9  
 Struts and ties (A, 1M)  
 Substitute mechanisms<sup>112,421g</sup>  
 Swivels (A, 1M)  
 Tape drives and devices (B, 3M)(B, 5M)<sup>21,189</sup>  
 Threads<sup>343</sup>  
 Three-dimensional drives;<sup>5,8,304</sup> Sec. 3.9  
 Time-measuring devices (C, 3M); timers<sup>274</sup>  
 Toggles<sup>138,144,275,427,498</sup>  
 Torsion devices<sup>141</sup>  
 Toys, mechanisms used in<sup>312</sup>  
 Tracks and rails (A, 1M)  
 Transducers (AB, 2M)(C, 3M)<sup>1,52,70,105,477</sup>  
 Transfer, of parts, or station advance (B, 2M)  
 Transfer, power to moving mechanisms (AB, 2M)  
 Transmissions and speed changers (BC, 5M);<sup>4,7,437</sup> see also Variable mechanical advantage and Speed-changing mechanisms  
 Transmissions, special (C, 4M)<sup>179</sup>  
 Tripping mechanisms (C, 4M)  
 Typewriting devices (C, 3M)<sup>192,266</sup>  
 Universal joints<sup>21,262,264,379,443</sup>  
 Valve gear (BC, 1M)  
 Valves (A, 1M); design of nonlinear<sup>335,374</sup>  
 Variable mechanical advantage and power devices (A, 1M);<sup>268,441,499</sup> see also Transmissions and speed changers  
 Washing devices (A, 1M)  
 Wedge devices (A, 3M)(B, 5M)  
 Weighing devices (AB, 1M)  
 Weights, for compensation and balancing (A, 1M)<sup>421f</sup>  
 Wheels (A, 1M)  
 Wheels, elastic (A, 1M)  
 Windmill and feathering devices (A, 1M)  
 Window-regulating mechanisms<sup>106,142,421f</sup>  
 Woodworking machines<sup>273</sup>

Writing instruments (C, 3M); see also Curve-drawing devices and Mathematical instruments

### 3.9 KINEMATIC PROPERTIES OF MECHANISMS

For a more complete literature survey, see Refs. 1 to 6 cited in Ref. 124, and the Engineering Index, currently available computer programs are listed in Ref. 129.

#### 3.9.1 The General Slider-Crank Chain (Fig. 3.43)

##### Nomenclature

$A$ = crankshaft axis	Block at $C$ = slider
$B$ = crankpin axis	$s$ = stroke
$C$ = wrist-pin axis	$\theta$ = crank angle
$FD$ = guide	$\phi$ = angle between connecting rod and slide, pressure angle
$AF = e$ = offset, $\perp FD$	$\tau = \angle ABC$
$AB = r$ = crank	$\eta = \angle BGP$ = auxiliary angle
$BC = l$ = connecting rod	$PG$ = collineation axis; $CP \perp FD$
$x$ = displacement of $C$ in direction of guide, measured from $F$	
$t$ = time	

The following mechanisms are derivable from the general slider-crank chain:

1. The *slider-crank mechanism*; guide fixed; if  $e \neq 0$ , called "offset," if  $e = 0$ , called "in-line";  $\lambda = r/l$ ; in case of the in-line slider crank, if  $\lambda < 1$ ,  $AB$  rotates; if  $\lambda > 1$ ,  $AB$  oscillates.
2. *Swinging-block mechanism*; connecting rod fixed; "offset" or "in-line" as in 1.
3. *Turning-block mechanism*; crank fixed; exact kinematic equivalent of 2; see Fig. 3.44.
4. The *standard geneva mechanism* is derivable from the special case,  $e = 0$  (see Fig. 3.45*b*).
5. Several *variations* of the *geneva mechanism* and other pin-and-slot or block-and-slot drives.

#### 3.9.2 The Offset Slider-Crank Mechanism (see Fig. 3.43 with $AFD$ stationary)

$$\text{Let} \quad \lambda = r/l \quad \epsilon = e/l \quad (3.40)$$

where  $l$  is the length of the connecting rod, then

$$s = l[(1 + \lambda)^2 - \epsilon^2]^{1/2} - l[(1 - \lambda)^2 - \epsilon^2]^{1/2} \quad (3.41)$$

$$\sin \phi = \epsilon + \lambda \sin \theta \quad (3.42)$$

$$\tau = \pi - \phi - \theta \quad (3.43)$$

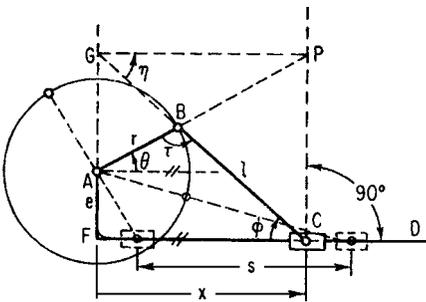


FIG. 3.43 General slider-crank chain.

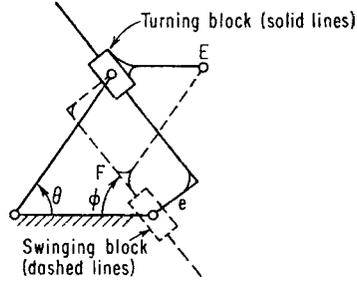


FIG. 3.44 Kinematic equivalence of the swinging-block and turning-block mechanisms, shown by redundant connection *EF*.

and 
$$x = r \cos \theta + l \cos \phi \tag{3.44}$$

Let the angular velocity of the crank be  $d\theta/dt = \omega$ ; then the slider velocity is given by

$$dx/dt = r\omega [ - \sin (\theta + \phi)/\cos \phi ] \tag{3.45}$$

Extreme value of  $dx/dt$  occurs when the auxiliary angle  $\eta = 90^\circ$ .<sup>116</sup>

Slider acceleration ( $\omega = \text{constant}$ ):

$$\frac{d^2x}{dt^2} = r\omega^2 \left[ \frac{- \cos (\theta + \phi)}{\cos \phi} - \lambda \frac{\cos^2 \theta}{\cos^3 \phi} \right] \tag{3.46}$$

Slider shock ( $\omega = \text{constant}$ ):

$$\frac{d^3x}{dt^3} = r\omega^3 \left[ \frac{\sin (\theta + \phi)}{\cos \phi} + \frac{3\lambda \cos \theta}{\cos^5 \phi} (\sin \theta \cos^2 \phi - \lambda \sin \phi \cos^2 \theta) \right] \tag{3.47}$$

For the angular motion of the connecting rod, let the angular velocity ratio,

$$m_1 = d\phi/d\theta = \lambda(\cos \theta/\cos \phi) \tag{3.48}$$

Then the angular velocity of the connecting rod

$$d\phi/dt = m_1\omega \tag{3.49}$$

Let 
$$m_2 = d^2\phi/d\theta^2 = m_1(m_1 \tan \phi - \tan \theta) \tag{3.50}$$

Then the angular acceleration of the connecting rod, at constant  $\omega$ , is given by

$$d^2\phi/dt^2 = m_2\omega^2 \tag{3.51}$$

In addition, let

$$m_3 = d^3\phi/d\theta^3 = 2m_1m_2 \tan \phi - m_2 \tan \theta + m_1^3 \sec^2 \phi - m_1 \sec^2 \theta \tag{3.52}$$

Then the angular shock of the connecting rod, at constant  $\omega$ , becomes

$$d^3\phi/dt^3 = m_3\omega^3$$

In general, the  $(n-1)$ th angular acceleration of the connecting rod, at constant,  $\omega$ , is given by

$$d^n\phi/dt^n = m_n\omega^n \tag{3.53}$$

where  $m_n = dm_{n-1}/d\theta$ . In a similar manner, the general expression for the  $(n - 1)$ th linear acceleration of the slider, at constant  $\omega$ , takes the form

$$d^n x/dt^n = r\omega^n M_n \tag{3.54}$$

where  $M_n = dM_{n-1}/d\theta$ ,

$$M_1 = -\sin(\theta + \phi)/\cos\phi \tag{3.55}$$

and  $M_2$  and  $M_3$  are the bracketed expressions in Eqs. (3.46) and (3.47), respectively.

Kinematic characteristics are governed by Eqs. (3.40) to (3.55). Examples for path and function generation, Ref. 432. Harmonic analyses, Refs. 39 and 296. Coupler curves, Ref. 104. Cognates, Ref. 182. Offset slider-crank mechanism can be used to reduce the friction of the slider in the guide during the “working” stroke; transmission-angle charts, Ref. 467.

Amplitudes of the harmonics are slightly higher than for the in-line slider-crank with the same  $\lambda$  value.

For a nearly constant slider velocity  $(1/\omega)(dx/dt) = k$  over a portion of the motion cycle, the proportions<sup>42</sup>

$$12k = 3e \pm \sqrt{9e^2 - 8(l^2 - 9r^2)}$$

may be useful.

### 3.9.3 The In-Line Slider-Crank Mechanism

( $e = 0$ )<sup>32,39,73,104,129,182,296,472,467</sup>

If  $\omega \neq$  constant, see Ref. 21. In general, see Eqs. (3.44) to (3.55).

Equations (3.56) and (3.57) give approximate values when  $\lambda < 1$ , and with  $\omega =$  constant. (For nomenclature refer to Fig. 3.43, with  $e = 0$ , and guide fixed.)

Slider velocity:  $dx/dt = r\omega(-\sin\theta - \frac{1}{2}\lambda\sin 2\theta)$  (3.56)

Slider acceleration:  $d^2x/dt^2 = r\omega^2(-\cos\theta - \lambda\cos 2\theta)$  (3.57)

**Extreme Values.**  $(dx/dt)_{\max}$  occurs when the auxiliary angle  $\eta = 90^\circ$ . For a prescribed extreme value,  $(1/r\omega)(dx/dt)_{\max}$ ,  $\lambda$  is obtainable from Eq. (22) of Ref. 116.

At extended dead center:  $d^2x/dt^2 = -r\omega^2(1 + \lambda)$  (3.58)

At folded dead center:  $d^2x/dt^2 = r\omega^2(1 - \lambda)$  (3.59)

Equations (3.58) and (3.59) yield exact extreme values whenever  $0.264 < \lambda < 0.88$ .<sup>472</sup>

**Computations.** See computer programs in Ref. 129, and also Kent’s “Mechanical Engineers Handbook,” 1956 ed., Sec. II, Power, Sec. 14, pp. 14-61 to 14-63, for displacements, velocities, and accelerations vs.  $\lambda$  and  $\theta$ ; similar tables, including also kinematics of connecting rod, are found in Ref. 73 for  $0.2 \leq \lambda \leq 0.7$  in increments of 0.1.

#### Harmonic Analysis<sup>39</sup>

$$x/r = A_0 + \cos\theta + \frac{1}{4}A_2\cos 2\theta - \frac{1}{16}A_4\cos 4\theta + \frac{1}{64}A_6\cos 6\theta - \dots \tag{3.60}$$

If  $\omega = \text{constant}$ ,

$$-(1/r\omega^2)(d^2x/dt^2) = \cos \theta + A_2 \cos 2\theta - A_4 \cos 4\theta + A_6 \cos 6\theta - \dots \quad (3.61)$$

where  $A_j$  are given in Table 3.1<sup>39</sup>

For harmonic analysis of  $\phi(\theta)$ , and for inclusion of terms for  $\omega \neq \text{constant}$ , see Ref. 39;

**TABLE 3.1** Values of  $A_j^*$

$l/r$	$A_2$	$A_4$	$A_6$
2.5	0.4173	0.0182	0.0009
3.0	0.3431	0.0101	0.0003
3.5	0.2918	0.0062	0.0001
4.0	0.2540	0.0041	0.0001
4.5	0.2250	0.0028	0.0000
5.0	0.2020	0.0021	
5.5	0.1833	0.0015	
6.0	0.1678	0.0012	

\*Biezeno, C. B., and R. Grammel, "Engineering Dynamics," (Trans. M. P. White), vol. 4, Blackie & Son, Ltd., London and Glasgow, 1954.

coupler curves (described by a point in the plane of the connecting rod) in Refs. 32 and 104; "cognate" slider-crank mechanism (i.e., one, a point of which describes the same coupler curve as the original slider-crank mechanism), Ref. 182; straight-line coupler-curve guidance, see VDI—Richtlinien No. 2136.

**3.9.4 Miscellaneous Mechanisms Based on the Slider-Crank Chain**

19,39,42,104,289,291,292,297,299,301,349,367,432,436

1. In-line swinging-block mechanism
2. In-line turning-block mechanism
3. External geneva motion
4. Shaper drive
5. Offset swinging-block mechanism
6. Offset turning-block mechanism
7. Elliptic slider-crank drive

For "in-line swinging-block" and "in-line turning block" mechanisms, see Fig. 3.45a and b. The following applies to both mechanisms.  $\lambda = r/a$ ;  $\theta$  is considered as input, with  $\omega_{AB} = \text{constant}$ .

Displacement: 
$$\phi = \tan^{-1} \frac{\lambda \sin \theta}{1 - \lambda \cos \theta} \quad (3.62)$$

Angular velocities (positive clockwise): 
$$\frac{d\phi}{dt} = \omega_{AB} \frac{\lambda \cos \theta - \lambda^2}{1 + \lambda^2 - 2\lambda \cos \theta} \quad (3.63)$$

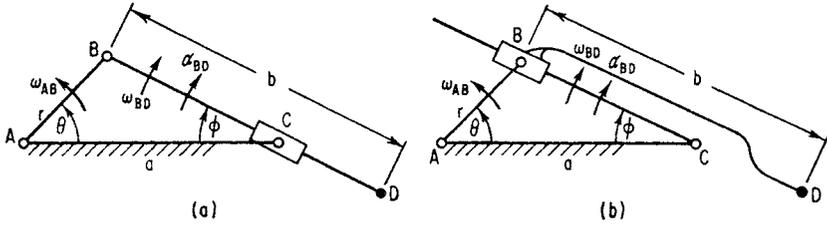


FIG. 3.45 (a) In-line swinging-block mechanism. (b) In-line turning-block mechanism.

$$\frac{1}{\omega_{AB}} \left| \frac{d\phi}{dt} \right|_{\min, \theta = 180^\circ} = \frac{-\lambda}{\lambda + 1} \tag{3.64}$$

$$\frac{1}{\omega_{AB}} \left( \frac{d\phi}{dt} \right)_{\max, \theta = 0^\circ} = \frac{\lambda}{1 - \lambda} \tag{3.65}$$

Angular acceleration: 
$$\alpha_{BD} = \frac{d^2\phi}{dt^2} = \frac{(\lambda^3 - \lambda) \sin \theta}{(1 + \lambda^2 - 2\lambda \cos \theta)^2} \omega_{AB}^2 \tag{3.66}$$

Extreme value of  $\alpha_{BD}$  occurs when  $\theta = \theta_{\max}$ , where

$$\cos \theta_{\max} = -G + (G^2 + 2)^{1/2} \tag{3.67}$$

and 
$$G = \frac{1}{4}(\lambda + 1/\lambda) \tag{3.68}$$

Angular velocity ratio  $\omega_{BD}/\omega_{AB}$  and the ratio  $\alpha_{BD}/\omega_{AB}^2$  are found from Eqs. (3.63) and (3.66), respectively, where  $\omega_{BD} = d\phi/dt$ . See also Sec. 3.9.7.

**Straight-Line Guidance.**<sup>42,467</sup> Point D (see Fig. 3.45a and b) will generate a close point-approximation to a straight line for a portion of its (bread-shaped) path, when

$$b = 3a - r + \sqrt{8a(a - r)} \tag{3.69}$$

**Approximate Circular Arc (for a portion of motion cycle).**<sup>42</sup> Point D (Fig. 3.45a and b) will generate an approximately circular arc whose center is at a distance c to the right of A (along AC) when

$$[b(a + c) - c(a - r)]^2 = 4bc(c - a)(a + r)$$

with  $b > 0$  and  $|c| > a > r$ .

Proportions can be used in intermittent drive by attachment of two additional links (VDI-Berichte, vol. 29, 1958, p. 28).<sup>42,473</sup>

**Harmonic Analysis**<sup>39,286,289,301</sup> (see Fig. 3.45a and b). Case 1,  $\lambda < 1$ :

$$\phi = \sum_{n=1}^{\infty} \frac{\lambda^n \sin n\theta}{n} \tag{3.70}$$

$$\frac{\omega_{BD}}{\omega_{AB}} = \frac{d\phi}{d\theta} = \sum_{n=1}^{\infty} \lambda^n \cos n\theta \tag{3.71}$$

Case 2,  $\lambda > 1$ :

$$\phi = \pi - \theta - \sum_{n=1}^{\infty} \frac{\sin n\theta}{n\lambda^n} \tag{3.72}$$

$$\frac{d\phi}{d\theta} = -1 - \sum_{n=1}^{\infty} \frac{\cos n\theta}{\lambda^n} \tag{3.73}$$

Note that in case 1  $AB$  rotates and  $BC$  oscillates, while in case 2 both links perform full rotations.

*External Geneva Motion.* Equations (3.62) to (3.68) apply. For more extensive data, including tables of third derivatives and various numerical values, see Intermittent-Motion Mechanisms, Sec. 3.9.7, and Refs. 252, 253, and 352.

An analysis of the “shaper drive” involving the turning-block mechanism is described in Ref. 289, part 2; see also Ref. 42.

*Offset Swinging-Block and Offset Turning-Block Mechanisms.*<sup>301</sup> (See Fig. 3.44.) Synthesis of offset turning-block mechanisms for path and function generation described in Ref. 432; see also Ref. 436 for velocities and accelerations; extreme values of angular velocity ratio  $d\phi/d\theta = q$  are related by the equation  $q_{\max}^{-1} + q_{\min}^{-1} = -2$ ; these occur for the same position of the driving link or for those whose crank angles add up to two right angles, depending on whether the driving link swings (oscillates) or rotates, respectively<sup>436</sup>; for graphical analysis of accelerations involving relative motion between two (instantaneously) coincident points on two moving links, use Coriolis’s acceleration, or complex numbers in analytical approach.<sup>106</sup>

For the “elliptic slider-crank drive” see Refs. 293, 295, and 297.

**3.9.5 Four-Bar Linkages (Plane)** (Refs. 2, 12, 16, 32, 42, 58, 61, 62, 104, 106, 115, 118, 122, 123, 125, 127–129, 159, 160, 166, 170, 173, 176, 180, 194, 199, 201, 205, 209, 210, 223, 249, 279, 284, 298, 300, 303, 307, 315, 380, 421f, 432, 435, 448, 453, 467, 471, 473, 476, 483, 489, 491)

See Fig. 3.46. Four-bar mechanism,  $ABCD$ ,  $E$  on coupler;  $AB =$  crank  $b$ ;  $BC =$  coupler  $c$ ;  $CD =$  crank or link  $d$ ;  $AD =$  fixed link  $a$ ;  $AB$  is assumed to be the driving link.

*Grashof’s Inequality.* Length of longest link + length of shortest link < sum of lengths of two intermediate links.

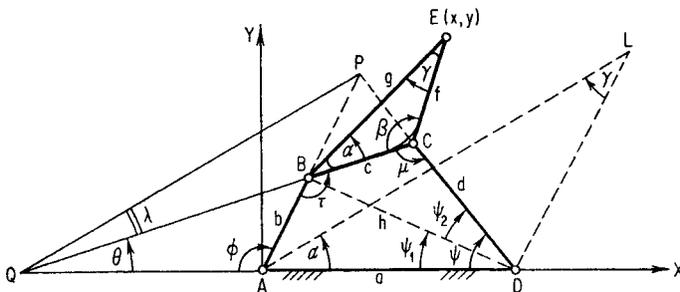


FIG. 3.46 Four-bar mechanism.

*Types of Mechanisms*

1. If Grashof's inequality is satisfied and  $b$  or  $d$  is the shortest link, the linkage is a "crank and rocker"; the shortest link is the "crank," and the opposite link is the "rocker."
2. If Grashof's inequality is satisfied and the fixed link is the shortest link, the linkage is a "drag linkage"; both cranks can make complete rotations.
3. All other cases except 4: the linkage is a "double-rocker" mechanism (cranks can only oscillate); this will be the case, for instance, whenever the coupler is the smallest link.
4. Special cases: where the equal sign applies in Grashof's inequality. These involve "folding" linkages and "branch positions," at which the motion is not positive. Example: parallelogram linkage; antiparallel equal-crank linkage ( $AB = CD, BC = AD$ , but  $AB$  is not parallel to  $CD$ ).<sup>207</sup>

**Angular Displacement.** In Fig. 3.46,  $\psi = \psi_1 + \psi_2$  (a minus sign would occur in front of  $\psi_2$  when a mechanism lies entirely on one side of diagonal  $BD$ ).

$$\psi = \cos^{-1} \frac{h^2 + a^2 - b^2}{2ah} + \cos^{-1} \frac{h^2 + d^2 - c^2}{2hd} \tag{3.74}$$

$$h^2 = a^2 + b^2 + 2ab \cos \phi \tag{3.75}$$

For alternative equation between  $\tan \frac{1}{2}\psi$  and  $\tan \frac{1}{2}\phi$  (useful for automatic computation) see Ref. 87.

The general closure equation:<sup>115</sup>

$$R_1 \cos \phi - R_2 \cos \psi + R_3 = \cos (\phi - \psi) \tag{3.76}$$

where  $R_1 = a/d \quad R_2 = ab \quad R_3 = (a^2 + b^2 - c^2 + d^2)/2bd$  (3.77)

The  $\phi, \tau$  equation:

$$p_1 \cos \phi - p_2 \cos \tau + p_3 = \cos (\phi - \tau) \tag{3.78}$$

where  $p_1 = b/c \quad p_2 = b/a \quad p_3 = (a^2 + b^2 + c^2 - d^2)/2ac$  (3.79)

$$\theta = \tau - \phi = \angle AQB \tag{3.80}$$

Extreme rocker-angle values in a crank and rocker:

$$\psi_{\max} = \cos^{-1} \{ [a^2 + d^2 - (b + c)^2] / 2ad \} \tag{3.81}$$

$$\psi_{\min} = \cos^{-1} \{ [a^2 + d^2 - (c - b)^2] / 2ad \} \tag{3.82}$$

Total range to rocker:  $\Delta\psi = \psi_{\max} - \psi_{\min}$

To determine inclination of the coupler  $\angle AQB = \theta$ , determine length of  $AC$ :

$$\overline{AC}^2 = a^2 + d^2 - 2ad \cos \psi = k^2 \quad (\text{say}) \tag{3.83}$$

Then compute  $\angle ABC = \tau$  from

$$\cos \tau = (b^2 + c^2 - k^2) / 2bc \tag{3.84}$$

and use Eq. (3.80). See also Refs. 307 and 448 for angular displacements; for extreme positions see Ref. 284; for geometrical construction of proportions for given ranges and extreme positions, see Ref. 173. For analysis of complex numbers see Ref. 106.

**Velocities.**

Angular velocity ratio:

$$\omega_{CD}/\omega_{AB} = QA/QD \tag{3.85}$$

Velocity ratio:

$$V_C/V_B = PC/PB \quad P = AB \times CD \quad V_p \text{ (} P \text{ on coupler)} = 0 \tag{3.86}$$

Velocity ratio of tracer point *E*:

$$V_E/V_B = PE/PB \tag{3.87}$$

Angular velocity ratio of coupler to input link:

$$\omega_{BC}/\omega_{AB} = BA/BP \tag{3.88}$$

When cranks are parallel, *B* and *C* have the same linear velocity, and  $\omega_{BC} = 0$ .

When coupler and fixed link are parallel,  $\omega_{CD}/\omega_{AB} = 1$ .

At an extreme value of angular velocity ratio,  $\lambda = 90^\circ$ .<sup>116,433</sup> When  $\omega_{BC}/\omega_{AB}$  is at a maximum or minimum,  $QP \perp CD$ .

Angular velocity ratio of output to input link is also obtainable by differentiation of Eq. (3.76):

$$m_1 = \frac{\omega_{CD}}{\omega_{AB}} = \frac{d\psi}{d\phi} = \frac{\sin(\phi - \psi) - R_1 \sin \phi}{\sin(\phi - \psi) - R_2 \sin \psi} \tag{3.89}$$

**Accelerations** ( $\omega_{AB} = \text{constant}$ ,  $t = \text{time}$ )

$$m_2 = \frac{d^2\psi}{d\phi^2} = \frac{1}{\omega_{AB}^2} \frac{d^2\psi}{dt^2} = \frac{(1 - m_1)^2 \cos(\phi - \psi) - R_1 \cos \phi + m_1^2 R_2 \cos \psi}{\sin(\phi - \psi) - R_2 \sin \psi} \tag{3.90}$$

Alternate formulation:

$$\frac{1}{\omega_{AB}^2} \frac{d^2\psi}{dt^2} = m_1(1 - m_1) \cot \lambda \tag{3.91}$$

(useful when  $m_1 \neq 1, 0$ , and  $\lambda \neq 0^\circ, 180^\circ$ ).

On extreme values, see Ref. 116; velocities, accelerations, and point-path curvature are discussed via complex numbers in Refs. 58, 106, and 427f; computer programs are in Ref. 129.

*Second Acceleration or Shock* ( $\omega_{AB} = \text{const}$ )

$$\begin{aligned} \frac{1}{\omega_{AB}^3} \frac{d^3\psi}{dt^3} &= \frac{d^3\psi}{d\phi^3} = m_3 \\ &= [R_1 \sin \phi - (m_1^3 \sin \psi - 3m_1 m_2 \cos \psi)R_2 - 3m_2(1 - m_1) \cos(\phi - \psi) \\ &\quad - (1 - m_1)^3 \sin(\phi - \psi)] / [\sin(\phi - \psi) - R_2 \sin \psi] \end{aligned} \tag{3.92}$$

*Coupler Motion.* Angular velocity of coupler:

$$d\theta/dt = (n_1 - 1)\omega_{AB}$$

where

$$n_1 = 1 + \frac{d\theta}{d\phi} = \frac{\sin(\phi - \tau) - p_1 \sin \phi}{\sin(\phi - \tau) - p_2 \sin \tau} \tag{3.93}$$

where  $p_1$  and  $p_2$  are as before [Eq. (3.79)]. Let

$$n_2 = \frac{d^2\theta}{d\phi^2} = \frac{(1 - n_1)^2 \cos(\phi - \tau) - p_1 \cos \phi + n_1^2 p_2 \cos \tau}{\sin(\phi - \tau) - p_2 \sin \tau} \tag{3.94}$$

Then the angular acceleration of the coupler,

$$d^2\theta/dt^2 = n_2 \omega_{AB}^2 \quad \omega_{AB} = \text{const} \tag{3.95}$$

If

$$\begin{aligned} n_3 &= \frac{d^3\theta}{d\phi^3} \\ &= [p_1 \sin \phi - (n_1^3 \sin \tau - 3n_1 n_2 \cos \tau) p_2 - 3n_2(1 - n_1) \cos(\phi - \tau) \\ &\quad - (1 - n_1)^3 \sin(\phi - \tau)] / [\sin(\phi - \tau) - p_2 \sin \tau] \end{aligned} \tag{3.96}$$

The angular shock of the coupler  $d^3\theta/dt^3$  at  $\omega_{AB} = \text{const}$  is given by

$$d^3\theta/dt^3 = n_3 \omega_{AB}^3 \tag{3.97}$$

See also Refs. 61 and 62 for angular acceleration and shock of coupler; for shock of points on the coupler see Ref. 298.

**Harmonic Analysis** ( $\psi$  vs.  $\phi$ ). Literature survey in Ref. 493. General equations for crank and rocker in Ref. 125. Formulas for special crank-and-rocker mechanisms designed to minimize higher harmonics:<sup>128</sup> Choose  $0^\circ \ll \nu \ll 90^\circ$ , and let

$$AB = \tan \frac{1}{2}\nu \quad BC = (1/\sqrt{2}) \sec \frac{1}{2}\nu = CD \quad AD = 1$$

$$\mu_{\max} = 90^\circ + \nu \quad \mu_{\min} = 90^\circ - \nu$$

$$\begin{aligned} \psi = \text{const} + \sum_{m=1}^{\infty} \frac{(-\tan \frac{1}{2}\nu)^m}{-m} \sin m\phi - \frac{C_0}{4} \sin \nu \cos \phi \\ - \sum_{m=1}^{\infty} \frac{\sin \nu}{4m} (C_{m-1} - C_{m+1}) \cos m\phi \end{aligned}$$

where letting

$$\sin \nu = p \quad a_2 = \frac{1}{4}p^2 \quad a_4 = \frac{3}{64}p^4 \quad a_6 = \frac{5}{512}p^6 \quad a_8 = \frac{35}{128^2}p^8$$

$$C_m \text{ (} m \text{ odd)} = 0$$

$$\text{and } C_0 = 1 + C_2 - C_4 + C_6 - C_8 + \dots \quad C_2 = a_2 + 4C_4 - 9C_6 + 16C_8 + \dots$$

$$C_4 = a_4 + 6C_6 - 20C_8 + \dots \quad C_6 = a_6 + 8C_8 + \dots$$

$$C_8 = a_8 + \dots$$

For numerical tables see Ref. 128. For four-bar linkages with adjacent equal links (driven crank = coupler), as in Ref. 128; see also Refs. 45, 300, 303.

**Three-Point Function Synthesis.** To find mechanism proportions when  $(\phi_p, \psi_i)$  are prescribed for  $i = 1, 2, 3$  (see Fig. 3.46).

$$a = 1 \quad b = \frac{w_2 w_3 - w_1 w_4}{w_1 w_6 - w_2 w_5} \quad d = \frac{w_2 w_3 - w_1 w_4}{w_3 w_6 - w_4 w_5}$$

$$c^2 = 1 + b^2 + d^2 - 2bd \cos(\phi_i - \psi_i) - 2d \cos \psi_i + 2b \cos \phi_i \quad i = 1, 2, 3$$

where

$$\begin{aligned} w_1 &= \cos \phi_1 - \cos \phi_2 & w_2 &= \cos \phi_1 - \cos \phi_3 \\ w_3 &= \cos \psi_1 - \cos \psi_2 & w_4 &= \cos \psi_1 - \cos \psi_3 \\ w_5 &= \cos(\phi_1 - \psi_1) - \cos(\phi_2 - \psi_2) & w_6 &= \cos(\phi_1 - \psi_1) - \cos(\phi_3 - \psi_3) \end{aligned}$$

**Four- and Five-Point Synthesis.** For maximum accuracy, use five points; for greater flexibility in choice of proportions and transmission-angle control, choose four points.

**Four-Point Path and Function Generation.** Path generation together with prescribed crank rotations in Refs. 106, 123, 371, and 421*f*. Function generation in Ref. 106, 381, and 421*f*.

**Five-Point Path and Function Generation.** See Refs. 123, 380, and 421*f*; the latter reference usable for five-point path vs. prescribed crank rotations, for Burmester point-pair determinations pertaining to five distinct positions of a plane, and for function generation with the aid of Ref. 127; additional references include 381, 435, and others at beginning of section; minimization of structural error in Refs. 16, 122, 249, and 421*f*; the latter with least squares; see Refs. 118, 122, and 194 for minimum-error function generators such as  $\log x$ ,  $\sin x$ ,  $\tan x$ ,  $e^x$ ,  $x^n$ ,  $\tanh x$ ; infinitesimal motions, Burmester points in Refs. 421*f*, 469, and 489.

**General.** Atlases for path generation (Ref. 199) and for function generation via "trace deviation" (Refs. 210 and 471); point-position-reduction discussed in Refs. 2, 106, 159, 194, 421*f*, and Sec. 3.5.7; nine-point path generation in Ref. 372.

**Coupler Curve.**<sup>32,104,315</sup> Traced by point  $E$ , in cartesian system with origin at  $A$ , and  $x$  and  $y$  axes as in Fig. 3.46:

$$U = f[(x - a) \cos \gamma + y \sin \gamma](x^2 + y^2 + g^2 - b^2) - gx[(x - a)^2 + y^2 + f^2 - d^2]$$

$$V = f[(x - a) \sin \gamma - y \cos \gamma](x^2 + y^2 + g^2 - b^2) + gy[(x - a)^2 + y^2 + f^2 - d^2]$$

$$W = 2gf \sin \gamma [x(x - a) + y^2 - ay \cot \gamma]$$

$$\text{With these} \quad U^2 + V^2 = W^2 \quad (3.98)$$

Equation (3.98) is a tricircular, trinodal, sextic, algebraic curve. Any intersection of this curve with circle through  $ADL$  (Fig. 3.46) is a double point, in special cases a cusp; coupler curves may possess up to three real double points or cusps (excluding curves traced by points on folding linkages); construction of coupler curves with cusps and application to instrument design (dwells, noiseless motion reversal, etc.) described in Refs. 32, 159, 279, theory in Ref. 63; detailed discussion of curves, including Watt straight-line motion and equality of two adjacent links, in Ref. 104; instant center (at intersection of cranks, produced if necessary) describes a cusp.

**Radius of Path Curvature  $R$  for Point  $E$**  (Fig. 3.46). In this case, as in other linkages, analytical determination of  $R$  is readily performed *parametrically*. Parametric equations of the coupler curve:

$$x = x(\phi) = -b \cos \phi + g \cos (\theta + \alpha) \tag{3.99}$$

$$y = y(\phi) = b \sin \phi + g \sin (\theta + \alpha) \tag{3.100}$$

where  $\theta = \tau - \phi$ ,  $\tau = \tau(\phi)$  is obtainable from Eqs. (3.74), (3.75), (3.83), and (3.84) and  $\cos \alpha = (g^2 - f^2 + c^2)/2gc$ .

$$x' = dx/d\phi = b \sin \phi - g(n_1 - 1) \sin (\theta + \alpha) \tag{3.101}$$

$$y' = dy/d\phi = b \cos \phi + g(n_1 - 1) \cos (\theta + \alpha) \tag{3.102}$$

$$x'' = d^2x/d\phi^2 = b \cos \phi - gn_2 \sin (\theta + \alpha) - g(n_1 - 1)^2 \cos (\theta + \alpha) \tag{3.103}$$

$$y'' = d^2y/d\phi^2 = -b \sin \phi + gn_2 \cos (\theta + \alpha) - g(n_1 - 1)^2 \sin (\theta + \alpha) \tag{3.104}$$

where  $n_1$  and  $n_2$  are given in Eqs. (3.93) and (3.94).

$$R = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} \tag{3.105}$$

**Equivalent or “Cognate” Four-Bar Linkages.** For Roberts’ theorem, see Sec. 3.6, Fig. 3.38. Proportions of the cognates are as follows (Figs. 3.38 and 3.46).

*Left cognate:*

$$\mathbf{AF} = \mathbf{BC}z \quad \mathbf{HF} = \mathbf{AB}z \quad \mathbf{HL} = \mathbf{CD}z \quad \mathbf{AL} = \mathbf{AD}z$$

where

$$z = (g/c)e^{i\alpha} \quad \alpha = \angle CBE$$

and where  $\mathbf{AF}$ , etc., represent the complex-number form of the vector  $\vec{AF}$ , etc.

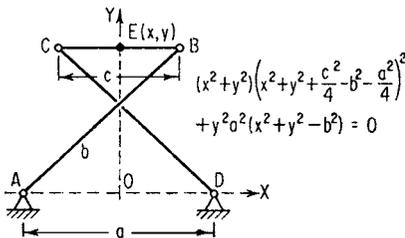
*Right cognate:*

$$\mathbf{GD} = \mathbf{BC}u \quad \mathbf{GK} = \mathbf{CD}u \quad \mathbf{LK} = \mathbf{AB}u \quad \mathbf{LD} = \mathbf{AD}u$$

where

$$u = (f/c)e^{-i\beta} \quad \beta = \angle ECB$$

The same construction can be studied systematically with the “Cayley diagram.”



**FIG. 3.47** Equal-crank linkage showing equation of symmetric coupler curve generated by point  $E$ , midway between  $C$  and  $B$ .

**Symmetrical Coupler Curves.** Coupler curves with an axis of symmetry are obtained when  $BC = CD = EC$  (Fig. 3.46); also by cognates of such linkages; used by K. Hunt for path of driving pin in geneva motions;<sup>201</sup> also for dwells and straight-line guidance (see Sec. 3.8, 5M). Symmetrical coupler-curve equation<sup>42</sup> for equal-crank linkage, traced by midpoint  $E$  of coupler in Fig. 3.47.

**Transmission Angles.** Angle  $\mu$  (Fig. 3.46) should be as close to  $90^\circ$  as possible; non-trivial extreme values occur when  $AB$  and

$AD$  are parallel or antiparallel ( $\phi = 0^\circ, 180^\circ$ ). Generally

$$\cos \mu = [(c^2 + d^2 - a^2 - b^2)/2cd] - (2ab/2cd) \cos \phi \tag{3.106}$$

$\cos \mu_{\max}$  occurs when  $\phi = 180^\circ$ ,  $\cos \phi = -1$ ;  $\cos \mu_{\min}$  when  $\phi = 0^\circ$ ;  $\cos \phi = 1$ . Good crank-and-rocker proportions are given in Sec. 3.6:

$$b^2 + a^2 = c^2 + d^2 \quad |\mu_{\min} - 90^\circ| = |\mu_{\max} - 90^\circ| = \nu \quad \sin \nu = ab/cd$$

A computer program for path generation with optimum transmission angles and proportions is described in Ref. 371. Charts for optimum transmission-angle designs are as follows: drag links in Ref. 160; double rockers in Ref. 166; general, in Refs. 170 and 205. See also VDI charts in Ref. 467.

**Approximately Constant Angular-Velocity Ratio of Cranks over a Portion of Crank Rotation** (see also Ref. 42). In Fig. 3.46, if  $d = 1$ , a three-point approximation is obtained when

$$a^2 = c^2 \frac{(1 - 2m_1)(m_1 - 2)}{9m_1} + b^2 \frac{(1 - m_1)(2 - m_1)}{m_1(m_1 + 1)} + \frac{(1 - m_1)(1 - 2m_1)}{(m_1 + 1)}$$

where the angular-velocity ratio  $m_1$  is given by Eq. (3.89). Useful only for limited crank rotations, possibly involving connection of distant shafts, high loads.

**Straight-Line Mechanisms.** Survey in Refs. 72, 208; modern and special applications in Refs. 223, 473, 476, theory and classical straight-line mechanisms in Ref. 42; see also below; order-approximation theory in Refs. 421f and 453.

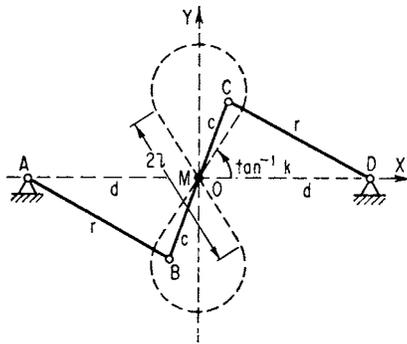


FIG. 3.48 Watt straight-line mechanism.

**Fifth-Order Approximate Straight Line via a Watt Mechanism.**<sup>42</sup> “Straight” line of length  $2l$ , generated by  $M$  on coupler, such that  $y \cong kx$  (Fig. 3.48).

Choose  $k, l, r$ ; let  $\beta = l(1 + k^2)^{-1/2}$ ; then maximum error from straight-line path  $\cong 0.038(1 + k^2)^3\beta^6$ . To compute  $d$  and  $c$ :

$$(d^2 - c^2) = [r^4 + 6(7 - 4\sqrt{3})l^4 + 3(3 - 2\sqrt{3})l^2r^2]^{1/2}$$

$$p_2 = 3(3 - 2\sqrt{3})\beta$$

$$4k^2d^2 = 2(1 + k^2)(d^2 - c^2 + r^2) + p_2(1 + k^2)^2$$

For less than fifth-order approximation, proportions can be simpler:  $AB = CD, BM = MC$ .

**Sixth-Order Straight Line via a Chebyshev Mechanism.**<sup>42,491</sup>  $M$  will describe an approximate horizontal straight line in the position shown in Fig. 3.49, when

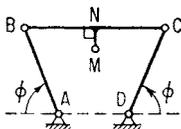


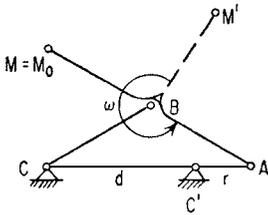
FIG. 3.49 Chebyshev straight-line mechanism.  $BN = NC = \frac{1}{2}AD = b$ ;  $AB = CD = r$ ;  $NM = C$  (+ downward). In general,  $\phi = 90^\circ - \theta$ , where  $\theta$  is the angle between axis of symmetry and crank in symmetry position.

$$a/r = (2 \cos^2 \phi \cos 2\phi) / \cos 3\phi$$

$$b/r = -\sin^2 2\phi / \cos 3\phi$$

$$c/r = (\cos^2 \phi \cos 2\phi \tan 3\phi) / \cos 3\phi$$

when  $\phi = 60^\circ$ ,  $NM = 0$ .



**Lambda Mechanism<sup>12,42</sup> and a Related Motion.** The four-bar lambda mechanism of Fig. 3.50 consists of crank  $AC' = r$ , fixed link  $CC' = d$ , coupler  $AB$ , driven link  $BC$ , with generating point  $M$  at the straight-line extension of the coupler, where  $BC = MB = BA = 1$ .  $M$  generates a symmetrical curve. In a related mechanism,  $M'B = BA$ ,  $\omega = \angle M'BA$  as shown, and  $M'$  generates another symmetrical coupler curve.

FIG. 3.50 Lambda mechanism.

*Case 1.* Either coupler curve of  $M$  contained between two concentric circles, center  $O_1$ ,  $O_1M_0C$  collinear.  $M = M_0$  when  $AC'C$  are collinear as shown.

Let  $\psi''$  be a parameter,  $0 \leq \psi'' \leq 45^\circ$ . Then a six-point approximate circle is generated by  $M$  with least maximum structural error when

$$r = 2 \sin \psi'' \sin 2\psi'' \sqrt{2 \cos 2\psi''} / \sin 3\psi''$$

$$d = \sin 2\psi'' / \sin 3\psi''$$

$$O_1C = 2 \cos^2 \psi'' / \sin 3\psi''$$

Radius  $R$  of generated circle (at precision points):

$$R = r \cot \psi''$$

Maximum radial (structural) error:

$$2 \cos 2\psi'' / \sin 3\psi''$$

For table of numerical values see Ref. 12.

*Case 2.* Entire coupler curve contained between two straight lines (six-point approximation of straight line with least maximum structural error). In the equations above,  $M'$  generates this curve when  $\angle M'BA = \omega = \pi + 2\psi''$ . Maximum deviation from straight line:<sup>12</sup>

$$2 \sin 2\psi'' \sqrt{2 \cos^3 2\psi''} / \sin 3\psi''$$

*Case 3.* Six-point straight line for a portion of the coupler curve of  $M'$ :

$$r = \frac{1}{4} \quad d = \frac{3}{4}$$

*Case 4.* Approximate circle for a portion of the coupler curve of  $M$ . Any proportions for  $r$  and  $d$  give reasonably good approximation to some circle because of symmetry. Exact proportions are shown in Refs. 12 and 42.

**Balancing of Four-Bar Linkages for High-Speed Operation.**<sup>421f,448</sup> Make links as light as possible; if necessary, counterbalance cranks, including appropriate fraction of coupler on each.

**Multilink Planar Mechanisms.** Geared five-bar mechanisms,<sup>95,119,120,421f</sup> six-link mechanisms.<sup>348,421f</sup>

**Design Charts.**<sup>467</sup> See also section on transmission angles.

**3.9.6 Three-Dimensional Mechanisms** (Refs. 33–37, 60, 75, 85, 86, 204, 224, 226, 250, 262, 294, 295, 302, 304, 305, 310, 327, 332, 361, 421f, 462–464, 466, 493, 495)

**Spherical Four-Bar Mechanisms**<sup>85</sup> ( $\theta_1$ , output vs.  $\theta_2$ , input) (Fig. 3.51)

$$A \sin \theta_1 + B \cos \theta_1 = C$$

where  $A = \sin \alpha_2 \sin \alpha_4 \sin \theta_2$  (3.107)

$$B = -\sin \alpha_4 (\sin \alpha_1 \cos \alpha_2 + \cos \alpha_1 \sin \alpha_2 \cos \theta_2)$$
 (3.108)

$$C = \cos \alpha_3 - \cos \alpha_4 (\cos \alpha_1 \cos \alpha_2 - \sin \alpha_1 \sin \alpha_2 \cos \theta_2)$$
 (3.109)

Other relations given in Ref. 85. Convenient equations between  $\tan \frac{1}{2}\theta_1$  and  $\tan \frac{1}{2}\theta_2$  given in Ref. 87.

Maximum angular velocity ratio  $d\theta_1/d\theta_2$  occurs when  $\lambda = 90^\circ$ .

Types of mechanisms: Assume  $\alpha_i$  ( $i = 1, 2, 3, 4$ )  $< 180^\circ$  and apply Grashof's rule (p. 3.51) to equivalent mechanism with identical axes of turning joints, such that all links except possibly the coupler,  $< 90^\circ$ .

Harmonic analysis: see Ref. 493.

**Special Cases of the Spherical Four-Bar Mechanism**

**Hooke's Joint.**  $\alpha_2 = \alpha_3 = \alpha_4 = 90^\circ$ ,  $90^\circ < \alpha_1 \leq 180^\circ$ ; in practice, if  $\alpha_1 = 180^\circ - \beta$ , then  $0 \leq \beta \leq 37\frac{1}{2}^\circ$ . If angles  $\theta$ ,  $\varphi$  ( $\theta = \theta_1 - 90^\circ$ ,  $\varphi = 180^\circ - \theta_2$ ) are measured from a starting position (shown in Fig. 3.52) in which the planes  $ABO$  and  $OC'D'$  are perpendicular, then

$$\tan \varphi / \tan \theta = \cos \beta \quad \beta = 180^\circ - \alpha_1$$

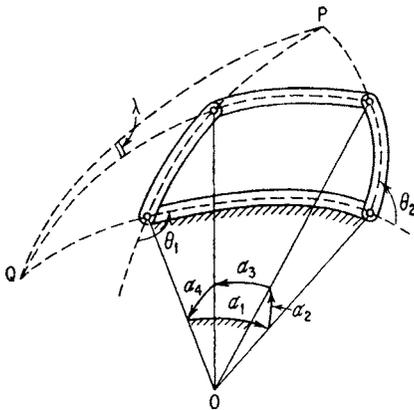


FIG. 3.51 Spherical four-bar mechanism.

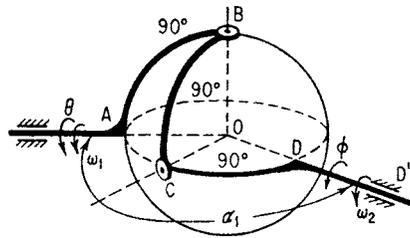


FIG. 3.52 Hooke's joint.

Angular velocity ratio:

$$\omega_2/\omega_1 = \cos \beta / (1 - \sin^2 \theta \sin^2 \beta)$$

Maximum value,  $(\cos \beta)^{-1}$ , occurs at  $\theta = 90^\circ, 270^\circ$ ; minimum value,  $\cos \beta$ , occurs at  $\theta = 0^\circ, 180^\circ$ .

A graph of  $\theta$  vs.  $\varphi$  will show two “waves” per revolution.

If  $\omega_1 = \text{constant}$ , the angular-acceleration ratio of shaft  $DD'$  is given by

$$\frac{1}{\omega_1^2} \frac{d^2\varphi}{dt^2} = \frac{\cos \beta \sin^2 \beta \sin 2\theta}{(1 - \sin^2 \beta \sin^2 \theta)^2} \quad \text{maximum at } \theta = \theta_{\max}$$

where  $\cos 2\theta_{\max} = G - (G^2 + 2)^{1/2}$  (3.110)

and  $G = (2 - \sin^2 \beta) / 2 \sin^2 \beta$  (3.111)

For  $d^2\varphi/dt^2$  as a function of  $\beta$ , see Ref. 264.

**Harmonic Analysis of Hooke's Joint.**<sup>264,493,495</sup> If  $\alpha_1 = 180^\circ - \beta$ , the amplitude  $\epsilon_m$  of the  $m$ th harmonic, in the expression  $\varphi(\theta)$ , is given by  $\epsilon_m = 0$ ,  $m$  odd; and  $\epsilon_m = (2/m)(\tan \frac{1}{2}\beta)^m$ ,  $m$  even. Two Hooke's joints in series<sup>96,327</sup> can be used to transmit constant (1:1) angular velocity ratio between two intersecting or nonintersecting shafts 1 and 3, provided that the angles between shafts 1 and 2, and the intermediate shaft 2 and 3, are the same (Fig. 3.53) and that when fork 1 lies in the plane of shafts 1 and 2, fork 2 lies in the plane of shafts 2 and 3; thus in case shafts 1 and 3 intersect, forks 1 and 2 are coplanar; see also Refs. 85 and 327 when

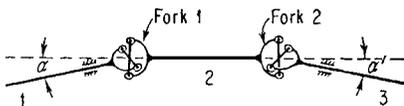


FIG. 3.53 Two Hooke's joints in series.

$\alpha \neq \alpha'$ , which may arise due to misalignment or the effect of manufacturing tolerances, or may be intentional for use as a vibration-excitation drive.

*Other Special Cases of the Spherical Four-Bar Mechanism*

*Wobble-plate mechanism:*  $\alpha_4 < 90^\circ, \alpha_2 = \alpha_3 = \alpha_1 = 90^\circ$ ; Ref. 35 gives displacements, velocities, and equation of “coupler curve.” See also Refs. 36, 294, and 493, the latter giving harmonic analysis.

*Two angles 90°, two angles arbitrary*<sup>36,295,305</sup>

*Fixed link = 90°:* Reference 332 includes applications to universal joints, gives velocities and accelerations (“space crank”).

*Spherical Four-Bar Mechanisms with Dwell.* See Ref. 35.

*Spherical Slider-Crank Mechanism.* Three turning pairs, one moving joint a turn slide, input pair and adjacent pair at right angles: see Ref. 304 for displacements; if input and output axes intersect at right angles, obtain “skewed Hooke's joint.”<sup>294,304</sup>

**Spatial Four-Bar Mechanisms** (see Fig. 3.42). To any spatial four-bar mechanism a corresponding spherical four-bar can be assigned as follows: Through  $O$  (Fig. 3.41) draw four radii, parallel to the axes of joints  $A, B, C,$  and  $D$  in Fig. 3.42 to intersect the surface of a sphere in four points corresponding to the joints of the spherical four-bar. The rotations of the spatial four-bar (Fig. 3.42) are the same as those of the corresponding spherical four-bar and are independent of the offsets  $a_{ij}$ , the minimum distances between the (noninteresting) axes  $i$  and  $j$  ( $ij = 12, 23, 34,$  and  $41$ ) of the spatial four-bar (Fig. 3.42). The input and output angles  $\theta_2, \theta_1$ , of the spatial four-bar, can be

measured as in Fig. 3.41 for the corresponding spherical four-bar; for  $s$ , the sliding at the output joint, measured from  $A$  to  $Q$  in Fig. 3.42, the general displacement equations<sup>85,86</sup> are (for rotations see Fig. 3.51 and accompanying equations)

$$s = \frac{A_1 \sin \theta_1 + B_1 \cos \theta_1 - C_1}{-A \cos \theta_1 + B \sin \theta_1} \quad (3.112)$$

where  $A$  and  $B$  are given in Eqs. (3.107) and (3.108) and

$$A_1 = (a_2 \cos \alpha_2 \sin \alpha_4 + a_4 \sin \alpha_2 \cos \alpha_4) \sin \theta_2 + s_2 \sin \alpha_2 \sin \alpha_4 \cos \theta_2 \quad (3.113)$$

$$\begin{aligned} B_1 = & -a_4 \cos \alpha_4 (\sin \alpha_1 \cos \alpha_2 + \cos \alpha_1 \sin \alpha_2 \cos \theta_2) \\ & -a_1 \sin \alpha_4 (\cos \alpha_1 \cos \alpha_2 - \sin \alpha_1 \sin \alpha_2 \cos \theta_2) \\ & + a_2 \sin \alpha_4 (\sin \alpha_1 \sin \alpha_2 - \cos \alpha_1 \cos \alpha_2 \cos \theta_2) \\ & + s_2 \sin \alpha_4 \cos \alpha_1 \sin \alpha_2 \sin \theta_2 \end{aligned} \quad (3.114)$$

$$\begin{aligned} C_1 = & -a_3 \sin \alpha_3 + a_4 \sin \alpha_4 (\cos \alpha_1 \cos \alpha_2 - \sin \alpha_1 \sin \alpha_2 \cos \theta_2) \\ & + a_1 \cos \alpha_4 (\sin \alpha_1 \cos \alpha_2 + \cos \alpha_1 \sin \alpha_2 \cos \theta_2) \\ & + a_2 \cos \alpha_4 (\cos \alpha_1 \sin \alpha_2 + \sin \alpha_1 \cos \alpha_2 \cos \theta_2) \\ & - s_2 \cos \alpha_4 \sin \alpha_1 \sin \alpha_2 \sin \theta_2 \end{aligned} \quad (3.115)$$

where  $a_1 = a_{12}$  of Fig. 3.42, and similarly  $a_2 = a_{23}$ ,  $a_3 = a_{34}$ ,  $a_4 = a_{41}$ ,  $\alpha_1 =$  angle between axes 1 and 2,  $\alpha_2 =$  angle between axes 2 and 3,  $\alpha_3 =$  angle between axes 3 and 4, and  $\alpha_4 =$  angle between axes 4 and 1 (Figs. 3.41 and 3.42). For complete nomenclature and sign convention, see Ref. 494. These equations are used principally in special cases, in which they simplify.

### *Special Cases of the Spatial Four-Bar and Related Three-Dimensional Four-Bar Mechanisms*<sup>8</sup>

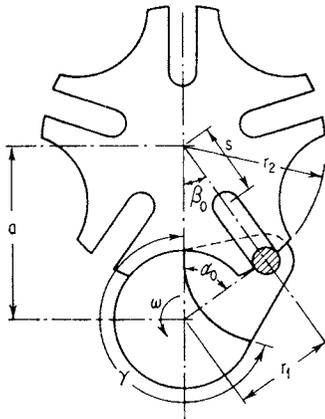
1. Spatial four-bar with two ball joints on coupler and two turning joints: displacements and velocities,<sup>60,201,310,361</sup> synthesis for function generation,<sup>86,250,361</sup> computer programs for displacements and synthesis according to Ref. 86 are listed in Ref. 129; forces and torques are listed in Ref. 60.
2. Spatial four-bar: three angles  $90^\circ$ .<sup>34,35</sup>
3. Spatial four-bar with one ball joint and two turn slides (three links).<sup>35</sup>
4. Spatial four-bar with one ball joint, one turn slide, two turning joints.<sup>462</sup>
5. The "3-D crank slide," one ball joint, one turn slide, intersecting axes; used for agitators.<sup>304</sup>
6. "Degenerate" mechanisms, wherein  $F = 1$ ,  $\sum f_i < 7$ ; conditions for,<sup>490</sup> practical constructions in;<sup>3</sup> see also Refs. 143, 194, and 464.
7. Spherical geneva.<sup>37,352</sup>
8. Spatial five-link mechanisms.<sup>6,93</sup>
9. Spatial six-link mechanisms.<sup>9,93</sup>

**Harmonic Analysis.** Rotations  $\theta_1$  vs.  $\theta_2$  in Ref. 493, which also includes special cases, such as Hooke's joint, wobble plates, and spherical-crank drive.

Special applications.<sup>35,36,294,304,332</sup>

Miscellaneous special shaft couplings;<sup>35,294</sup> Cayley-Klein parameters and dual vectors;<sup>85</sup> screw axes and graphical methods.<sup>36,151,196,462</sup>

**3.9.7 Intermittent-Motion Mechanisms** (Refs. 11, 16, 21, 22, 44, 148, 149, 195, 201, 204, 213, 214, 227, 252, 253, 307, 351, 352, 355, 377, 424, 425, 426, 429, 442, 450, 451, 466, 468)



**FIG. 3.54** External Geneva mechanism in starting position.

The external Geneva<sup>252,253,352</sup> is an intermittent-motion mechanism. In Fig. 3.54,

- $a$  = center distance
- $\alpha$  = angle of driver, radians ( $|\alpha| = \alpha_0$ )
- $\beta$  = angle of driven or Geneva wheel, radians [ $\alpha$  and  $\beta$  measured from center-line in the direction of motions ( $|\beta| = \beta_0$ )]
- $r_1$  = radius to center of driving pin
- $\gamma$  = locking angle of driver, radians
- $n$  = number of equally spaced slots in Geneva ( $\geq 3$ ) at start:  $|\alpha| = \alpha_0$ ,  $|\beta| = \beta_0 = \pi/2 - \alpha_0$  so that driving pin can enter slot tangentially to reduce shock
- $r_2$  = radius of Geneva =  $a \cos \beta_0$
- $r'_2$  = outside radius of Geneva wheel, with correction for finite pin diameter<sup>351</sup>  
 $= r_2 \sqrt{1 + r_p^2/r_2^2}$ ;  $r_p$  = pin radius

$$\epsilon = \text{gear ratio} = \frac{\text{angle moved by driver during motion}}{\text{angle moved by Geneva during motion}} = \frac{n - 2}{2}$$

$$\mu = \text{radius ratio} = r_2/r_1 = \cot \beta_0$$

$$s = \text{distance of center of semicircular end of slot from center of Geneva} \leq a(1 - \sin \beta_0)$$

$$\gamma = \text{angle of locking action} = (\pi/n)(n + 2) \text{ rad; note that classical locking action shown is subject to play in practice; better constructions in 4M, Sec. 3.8}$$

$$v = \text{ratio, time of motion of Geneva wheel to time for one revolution of driver} = (n - 2)/2n (< 1/2)$$

$$r_1 = a \sin \beta_0, \beta_0 = \pi/n$$

Let  $\omega$  = angular velocity of driving wheel, assumed constant,  $t$  = time.

**Displacement ( $\beta$  vs.  $\alpha$ ).** Let  $r_1/a = \lambda$ ; then

$$\beta = \tan^{-1} [\lambda \sin \alpha / (1 - \lambda \cos \alpha)] \tag{3.116}$$

**Velocities**

$$\frac{1}{\omega} \frac{d\beta}{dt} = \frac{\lambda \cos \alpha - \lambda^2}{1 - 2\lambda \cos \alpha + \lambda^2} \tag{3.117}$$

$$\frac{1}{\omega} \left( \frac{d\beta}{dt} \right)_{\max} = \frac{\lambda}{1 - \lambda} \quad \text{at } \alpha = 0 \tag{3.118}$$

**Accelerations**

$$\frac{1}{\omega^2} \frac{d^2\beta}{dt^2} = \frac{(\lambda^3 - \lambda) \sin \alpha}{(1 + \lambda^2 - 2\lambda \cos \alpha)^2} \tag{3.119}$$

$$(1/\omega^2)(d^2\beta/dt^2)_{\text{initial}\alpha = -\alpha_0} = \tan \beta_0 = r_1/r_2 \tag{3.120}$$

Maximum acceleration occurs at  $\alpha = \alpha_{\text{max}}$ , where

$$\cos \alpha_{\text{max}} = -G + (G^2 + 2)^{1/2} \quad G = 1/4(\lambda + 1/\lambda) \tag{3.121}$$

**Second Acceleration or Shock**

$$\frac{1}{\omega^3} \frac{d^3\beta}{dt^3} = \frac{\lambda(\lambda^2 - 1)[2\lambda \cos^2 \alpha + (1 + \lambda^2) \cos \alpha - 4\lambda]}{(1 + \lambda^2 - 2\lambda \cos \alpha)^3} \tag{3.122}$$

$$\frac{1}{\omega^3} \frac{d^3\beta}{dt^3} (\alpha = 0) = \frac{\lambda(\lambda + 1)}{(\lambda - 1)^3} \tag{3.123}$$

**Starting Shock** ( $|\beta| = \beta_0$ )

$$\frac{1}{\omega^3} \frac{d^3\beta}{dt^3} = \frac{3\lambda^2}{1 - \lambda^2}$$

**Design Procedure** ( $\omega = \text{const}$ )

1. Select number of stations ( $n \geq 3$ ).
2. Select center distance  $a$ .
3. Compute:  $r_1 = a \sin \beta_0$ ;  $r_2'^2 = (a \cos \beta_0)^2 + r_p^2$ ;  $s \leq a(1 - \sin \beta_0)$ ;

$$|\alpha_0| = (\pi/2)[(n - 2)/n] \quad \rho|\beta_0| = \pi/2 - \alpha_0 \quad \text{rad}$$

$$\gamma = (\pi/n)(n + 2) \quad \text{rad}$$

4. Determine kinematic characteristics from tables, including maximum velocity, acceleration, and shock:  $d^n\beta/dt^n = \omega^n d^n\beta/d\alpha^n$ , taking the last fraction from the tables.

Check for resulting forces, stresses, and vibrations. See Tables 3.2 and 3.3.

**TABLE 3.2** External Geneva Characteristics<sup>253\*</sup>

$n$	$\beta_0$	$\alpha_0$	$\epsilon$	$(r_1/a)$	$(r_2/a)$	$\mu$	$s_{\text{min}}/a$	$\gamma$	$\nu$	$(d\beta/d\alpha)_{\text{max}}$
3	60°	30°	0.5	0.8660	0.5000	0.5774	0.13397	300°	0.1667	6.46
4	45°	45°	1	0.7071	0.7071	1.0000	0.2929	270°	0.2500	2.41
5	36°	54°	1.5	0.5878	0.8090	1.3764	0.4122	252°	0.3000	1.43
6	30°	60°	2	0.5000	0.8660	1.7320	0.5000	240°	0.3333	1.00
7	25°43'	64°17'	2.5	0.4339	0.9009	2.0765	0.5661	231°26'	0.3571	0.766
8	22°30'	67°30'	3	0.3827	0.9239	2.4142	0.6173	225°	0.3750	0.620
9	20°	70°	3.5	0.3420	0.9397	2.7475	0.6580	220°	0.3889	0.520
10	18°	72°	4	0.3090	0.9511	3.0777	0.6910	216°	0.4000	0.447
$\infty$	0°	90°	$\infty$	0	1	$\infty$	1	180°	0.5000	0

\*O. Lichtwitz, Getriebe fuer Aussetzende Bewegung, Springer-Verlag OHG, Berlin, 1953.

**TABLE 3.3** External Geneva Characteristics<sup>253\*</sup>

$n$	$(d^2\beta/d\alpha^2)_{\text{initial}}$ $\alpha = -\alpha_0$	$\alpha_{\text{max}}$	$(d^2\beta/d\alpha^2)_{\text{max}}$	$(d^3\beta/d\alpha^3)_{\alpha=0}$
3	1.732	4°46'	31.44	-672
4	1.000	11°24'	5.409	-48.04
5	0.7265	17°34'	2.299	-13.32
6	0.5774	22°54'	1.350	-6.000
7	0.4816	27°33'	0.9284	-3.429
8	0.4142	31°38'	0.6998	-2.249
9	0.3640	35°16'	0.5591	-1.611
10	0.3249	38°30'	0.4648	-1.236
$\infty$	0	90°	0	0

\*O. Lichtwitz, *Getriebe fuer Aussetzende Bewegung*, Springer-Verlag OHG, Berlin, 1953.

**Modifications of Standard External Geneva.** More than one driving pin;<sup>252,253</sup> pins not equally spaced;<sup>16,252,253</sup> designs for all small indexing mechanisms;<sup>355</sup> for high-speed indexing;<sup>213</sup> mounting driving pin on a planet pinion to reduce peak loading;<sup>11,253,377</sup> pin guided on four-bar coupler<sup>201</sup> (see also Sec. 3.8, Ref. 5M); double rollers and different entrance and exit slots, especially for starwheels;<sup>227,466</sup> eccentric gear drive for pin.<sup>204</sup>

**Internal Genevas.**<sup>252,253,352</sup> Used when  $\nu > 1/2$ ; better kinematic characteristics, but more expensive.

**Star Wheels.**<sup>227,252,253,466</sup> Both internal and external are used; permits considerable freedom in choice of  $\nu$ , which can equal unity, in contrast to genevas. Kinematic properties of external star wheels are better or worse than of external geneva with same  $n$ , according as the number of stations (or shoes),  $n$ , is less than six or greater than five, respectively.

**Special Intermittent and/or Dwell Linkages.** The three-gear drive<sup>21,114,195,215,424,442</sup> cardioid drive (slotted link driven by pin on planetary pinion);<sup>352,425,426</sup> link-gear (and/or) -cam mechanisms to produce dwell, reversal, or intermittent motions<sup>22,449,450,451</sup> include link-dwell mechanisms;<sup>148</sup> eccentric-gear mechanisms.<sup>149</sup> These special motions may be required when control of rest, reversal, and kinematic characteristics exceeds that possible with the standard genevas.

**3.9.8 Noncircular Cylindrical Gearing and Rolling-Contact Mechanisms**

(Refs. 16, 43, 76, 117, 255, 256, 292, 314, 318, 319, 326, 341, 350, 356, 429, 438, 483)

Most of the data for this article are based on Refs. 43 and 318. Noncircular gears can be used for producing positive unidirectional motion; if the pitch curves are closed curves, unlimited rotations may be possible; only externally meshed, plane spur-type gearing will be considered; point of contact between pitch surfaces must lie on the line of centers.

A pair of roll curves may serve as pitch curves for noncircular gears (see Table 3.4):  $C$  = center distance;  $\beta$  = angle between the common normal to roll-curves at contact and the line of centers. Angular velocities  $\omega_1$  and  $\omega_2$  measured in opposite directions; polar-coordinate equations of curves;  $R_1 = R_1(\theta_1)$ ,  $R_2 = R_2(\theta_2)$ , such that the points  $R_1(\theta_1 = 0)$ ,  $R_2(\theta_2 = 0)$  are in mutual contact, where  $\theta_1$  and  $\theta_2$  are respective

oppositely directed rotations from a starting position. Centers  $O_1$ ,  $O_2$  and contact point  $Q$  are collinear.

**Twin Rolling Curves.** Mating or pure rolling of two identical curves, e.g., two ellipses when pinned at foci.

**Mirror Rolling Curves.** Curve mates with mirror image.

### Theorems

1. Every mating curve to a mirror (twin) rolling curve is itself a mirror (twin) rolling curve.
2. All mating curves to a given mirror (twin) rolling curve will mate with each other.
3. A closed roll curve can generally mate with an entire set of different closed roll curves at varying center distances, depending upon the value of the "average gear ratio."

**Average Gear Ratio.** For mating closed roll curves: ratio of the total number of teeth on each gear.

**Rolling Ellipses and Derived Forms.** If an ellipse, pivoted at the focus, mates with a roll curve so that the average gear ratio  $n$  (ratio of number of teeth on mating curve to number of teeth on ellipse) is integral, the mating curve is called an " $n$ th-order ellipse." The case  $n = 1$  represents an identical (twin) ellipse; second-order ellipses are oval-shaped and appear similar to ordinary ellipses; third-order ellipses appear pear-shaped with three lobes; fourth-order ellipses appear nearly square;  $n$ th-order ellipses appear approximately like  $n$ -sided polygons. Equations for several of these are found in Ref. 43. Characteristics for five noncircular gear systems are given in Table 3.4.<sup>43</sup>

**Design Data.**<sup>43</sup> Data are usually given in one of three ways:

1. Given  $R_1 = R_1(\theta_1)$ ,  $C$ . Find  $R_2$  in parametric form:  $R_2 = R_2(\theta_1)$ ;  $\theta_2 = \theta_2(\theta_1)$ .

$$\theta_2 = -\theta_1 + C \int_0^{\theta_1} \frac{d\theta_1}{C - R_1(\theta_1)} \quad R_2 = C - R_1(\theta_1)$$

2. Given  $\theta_2 = f(\theta_1)$ ,  $C$ . Find  $R_1 = R_1(\theta_1)$ ,  $R_2 = R_2(\theta_2)$ .

$$R_1 = \frac{(df/d\theta_1)C}{1 + df/d\theta_1} \quad R_2 = C - R_1$$

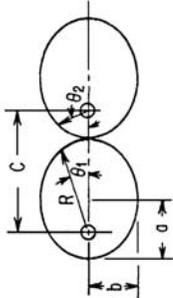
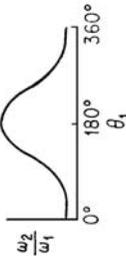
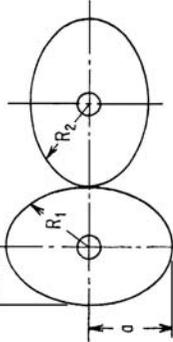
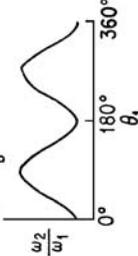
3. Given  $\omega_2/\omega_1 = g(\theta_1)$ ,  $C$ . Find  $R_1 = R_1(\theta_1)$ ,  $R_2 = R_2(\theta_2)$ .

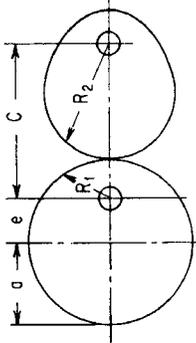
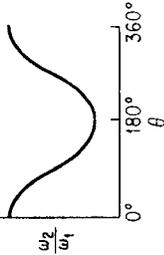
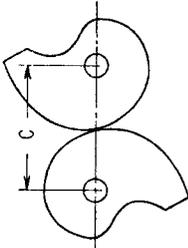
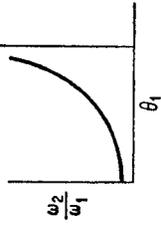
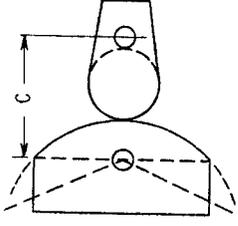
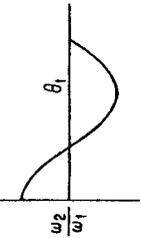
$$R_1 = \frac{Cg(\theta_1)}{1 + g(\theta_1)} \quad R_2 = C - R_1 \quad \theta_2 = \int_0^{\theta_1} g(\theta_1) d\theta_1$$

4. **Checking for closed curves:** Let  $R_1 = R_1(\theta_1)$  be a single-turn closed curve; then  $R_2 = R_2(\theta_2)$  will be a single-turn closed curve also, if and only if  $C$  is determined from

$$4\pi = C \int_0^{2\pi} \frac{d\theta_1}{C - R_1(\theta_1)}$$

TABLE 3.4 Characteristics of Five Noncircular Gear Systems<sup>43</sup>

Type	Comments	Basic equations	Velocity equation
 <p>Two ellipses pivoted at foci</p>	<p>Gears are identical, easy to manufacture. Used for quick-return mechanisms, printing presses, automatic machinery.</p>	$R = \frac{b^2}{a(1 + \epsilon \cos \theta)}$ <p>assuming <math>\theta = 0</math> at <math>R = R_{\min}</math>  <math>\epsilon =</math> eccentricity  <math>a = \sqrt{1 - (b/a)^2}</math>  <math>a = \frac{1}{2}</math> major axis  <math>b = \frac{1}{2}</math> minor axis</p>	$\omega_2 = \omega_1 \left[ \frac{r^2 + 1 + (r^2 - 1) \cos 2\theta_2}{2r} \right]$ <p>where <math>r = \frac{R_{\max}}{R_{\min}}</math></p> 
 <p>Second-order ellipses rotated about their geometric centers</p>	<p>Gears are identical. Well-known geometry. Better balance than true elliptical. Two complete speed cycles in one revolution.</p>	$R = \frac{2ab}{(a + b) - (a - b) \cos 2\theta}$ <p>assuming <math>\theta = 0</math> at <math>R = R_{\min}</math>  <math>C = a + b</math>  <math>a =</math> max radius  <math>b =</math> min radius</p>	$\omega_2 = \omega_1 \left[ \frac{r + 1 + (r^2 - 1) \cos 2\theta_2}{2r} \right]$ <p>where <math>r = \frac{a}{b}</math></p> 

 <p>Eccentric circular gear meshing with its conjugate</p>	<p>Standard circular spur gear can be employed as the eccentric. Mating gear has special shape.</p>	$R_1^2 - 2eR_1 \cos \theta_1 = a^2 - e^2$ $\theta_2 = \theta_1 + C \int_0^{\theta_1} \frac{d\theta_1}{C - R_1}$ $C = R_1 + R_2$	 $\frac{\omega_2}{\omega_1} = \frac{R_1}{C - R_1}$
 <p>Logarithmic spirals</p>	<p>Gears can be identical or in combinations to give variety of functions. Must be open gears.</p>	$R_1 = A e^{k\theta_1}$ $R_2 = C - R_1 = A e^{k\theta_2}$ $\theta_2 = \frac{1}{k} [\log(C - A e^{k\theta_1}) - \log A]$ <p><math>e</math> = base of natural logarithm  <math>A</math> = radius at <math>\theta = 0</math></p>	 $\frac{\omega_2}{\omega_1} = \frac{R_1}{C - R_1}$
 <p>Sine-function gears</p>	<p>For producing angular displacement proportional to sine of input angle. Must be open gears.</p>	$\theta_2 = k \sin \theta_1$ $R_2 = \frac{C}{1 + k \cos \theta_1}$ $R_1 = C - R_2$	 $\frac{\omega_2}{\omega_1} = k \cos \theta_1$

When the average gear ratio is not unity, see Ref. 43.

5. *Checking angle*  $\beta$ , also called the angle of obliquity:

$$\beta = \tan^{-1} \frac{1}{R_i} \frac{dR_i}{d\theta_i} \quad i = 1 \text{ or } 2.$$

Values of  $\beta$  between 0 and 45° are generally considered reasonable.

6. *Checking for tooth undercut*: Let  $\rho$  = radius of curvature of pitch curve (roll curve)

$$\rho = \frac{[R_i^2 + (dR_i/d\theta_i)^2]^{3/2}}{R_i^2 - R_i(d^2R_i/d\theta_i^2) + 2(dR_i/d\theta_i)^2} \quad i = 1 \text{ or } 2$$

Number of teeth:

$$T_{\min} = \begin{cases} 32 & \text{for } 14\frac{1}{2}^\circ \text{ pressure angle cutting tool} \\ 18 & \text{for } 20^\circ \text{ pressure angle cutting tool} \end{cases}$$

Condition to avoid undercut in noncircular gears:

$$\rho > \frac{T_{\min}}{2 \times \text{diametral pitch}}$$

7. *Determining length S of roll curves*:

$$S = \int_0^{2\pi} \left[ R^2 + \left( \frac{dR}{d\theta} \right)^2 \right]^{1/2} d\theta$$

This is best computed automatically by numerical integration or determined graphically by large-scale layout.

8. *Check on number of teeth*: For closed single-turn curves,

$$\text{Number of teeth} = (S \times \text{diametral pitch})/\pi$$

Diametral pitch should be integral but may vary by a few percentage points. For symmetrical twin curves, use odd number of teeth for proper meshing following identical machining.

Manufacturing information in Ref. 43.

*Special Topics in Noncircular Gearing.* Survey;<sup>341</sup> elliptic gears;<sup>292,350,356</sup> non-circular cams and rolling-contact mechanisms, such as in shears and recording instruments;<sup>117,256,314</sup> noncircular bevel gears;<sup>319</sup> algebraic properties of roll curves;<sup>483</sup> miscellaneous.<sup>255,326,438</sup>

### 3.9.9 Gear-Linked-Cam Combinations and Miscellaneous Mechanisms<sup>396</sup>

Two-gear drives;<sup>287,391,421f,474,475</sup> straight-type mechanisms in which rack on slide drives output gear (Refs. 285, 330, 390); mechanical analog computing mechanisms;<sup>21,40,137,216,306,328,338,446</sup> three-link screw mechanisms;<sup>359</sup> ratchets;<sup>18,362,370</sup> function generators with two four-bars in series;<sup>232,248</sup> two-degree-of-freedom computing mechanisms;<sup>328</sup> gear-train calculations;<sup>21,24,178,280,282,360</sup> the harmonic drive;<sup>68,126,316</sup> design of variable-speed drives;<sup>31</sup> rubber-covered rollers;<sup>382</sup> eccentric-gear drives.<sup>152</sup>

### 3.9.10 Robots and Manipulators<sup>97,102,156,187,240,320,373,421f,444,454,470</sup>

Robots are used for production, assembly, materials handling, and other purposes. Mechanically most robots consist of computer-controlled, joint-actuated, open kinematic chains terminating in an “end effector,” such as a gripper, hand, or a tool adaptor, which is used for motion transfer. The gripper may have many degrees of freedom, just as does the human hand. The mechanical design of robot mechanisms can include both rigid and elastic elements and involves the determination of kinematic structure, ranges of motion, useful work space, dexterity, kinematics, joint actuation, mechanical advantage, dynamics, power requirements, optimization, and integration with the electronic and computer portions of the robot. A general survey can be found in Heer<sup>187</sup> and Roth,<sup>373</sup> while more specialized investigation can be found in Refs. 97, 102, 156, 240, 320, 421f, 444, 454, and 470. The subject is extensive and continuously expanding.

### 3.9.11 Hard Automation Mechanisms<sup>421c,421f</sup>

For highly repetitive spatial automation tasks, robotic devices with their multiple programmed inputs are greatly “over-qualified.” For these tasks, single-input, purely mechanical spatial mechanisms can be more economical and efficient. For design-synthesis of these see Refs. 421c and 421f.

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